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Vector bundle moduli and small instanton transitions

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Abstract: We give the general presciption for calculating the moduli of irreducible, stable SU(n) holomorphic vector bundles with positive spectral covers over elliptically fibered Calabi-Yau threefolds. Explicit results are presented for Hirzebruch base surfaces $B = \mathbb{F}_r$. The transition moduli that are produced by chirality changing small instanton phase transitions are defined and specifically enumerated. The origin of these moduli, as the deformations of the spectral cover restricted to the "lift" of the horizontal curve of the $M5$-brane, is discussed. We present an alternative description of the transition moduli as the sections of rank n holomorphic vector bundles over the $M5$-brane curve and give explicit examples. Vector bundle moduli appear as gauge singlet scalar fields in the effective low-energy actions of heterotic superstrings and heterotic M-theory.

Keywords: Differential and Algebraic Geometry, Superstring Vacua, M-Theory.

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Contents

1. Introduction

G-instantons, that is, solutions of the hermitean Yang-Mills equations with structure group $G \subset E_8$ on complex manifolds, play a prominent role in determining the vacuum structure of both the weakly coupled heterotic superstring [1] (see also [2]) and heterotic M-theory [3, 4, 5]. Unfortunately, for manifolds of dimension greater than one, it has proven difficult or impossible to find the explicit one-form gauge connections that solve these equations. However, it was demonstrated by Donaldson [6] and Uhlenbeck and Yau [7] that there is a one-to-one relationship between the solutions of the hermitean Yang-Mills equations and the construction of stable, holomorphic vector bundles with structure group G over the same complex manifold. Happily, construction of such vector bundles is relatively straightforward, as was demonstrated for elliptically fibered Calabi-Yau manifolds by Friedman, Morgan and Witten [8] and Donagi [9]. Hence, to construct vacua of heterotic superstrings and M-theory, one can either attempt to solve the hermitean Yang-Mills equations or, more simply, construct the associated stable, holomorphic vector bundles. This latter approach has been used extensively in [10, 11, 12] to study charged matter in heterotic compactifications and in [13, 14, 15] to search for phenomenologically relevant grand unified or standard model vacua in heterotic M-theory.

Any solution of the hermitean Yang-Mills equations will depend on a number of independent integration constants, the "instanton moduli". The number and structure of these moduli will, in turn, depend on the geometry of the complex manifold, on the chosen structure group of the instanton and on the type of solution itself. When heterotic superstrings or M-theory are compactified to lower dimension on a complex manifold, instanton moduli will appear, not as constants, but as independent, gauge singlet scalar fields on the uncompactified space. These moduli fields, although relatively little explored, play a substantial role in the physics of both heterotic string theory and heterotic M-theory. First, we note that superstrings wrapped on holomorphic curves in the complex manifold will induce a non-perturbative contribution to the superpotential of many of the compactification moduli, including the instanton moduli [16, 17]. Computations of the superpotential for both Calabi-Yau and $M5$-brane moduli have been presented in [18]–[22]. We will give a detailed calculation of the superpotential for instanton moduli in a subsequent publication [23]. Suffice it here to say that this superpotential mixes instanton moduli with other geometric moduli. It follows that including instanton moduli, and computing their superpotential, is essential for the over-all study of the stability of heterotic superstring and M-theory vacua. Secondly, instanton moduli also play an important role in phenomenology and cosmology [24, 25, 26]. For example, in the recently proposed Ekpyrotic scenarios of the early universe [27]–[31], M-brane moduli interact with and transfer energy to instanton moduli, both before, during and after the cataclysmic collision that produces the "big bang". Hence, much of the structure of primordial fluctuations, reheating and the value of the cosmological constant, for example, depend on the theory of the instanton moduli.

As stated above, it is generically unknown how to compute explicit solutions of the hermitean Yang-Mills equations on complex manifolds. Hence, the number and properties of instanton moduli are difficult to determine from this point of view. However, the theorems of Donaldson, Uhlenbeck and Yau allow one to approach this issue from the point of view of the associated stable, holomorphic vector bundles. Here, one is much more successful. Specifically, we will show, within the context of irreducible, stable, holomorphic $\mathrm{SU}(n)$ vector bundles with positive spectral covers over elliptically fibered Calabi-Yau threefolds, how to compute the the precise number of "vector bundle moduli" and, in the process, exhibit their origin and some of their properties. These vector bundle moduli are, of course, identical to the instanton moduli of the associated Yang-Mills connection. We will, henceforth, refer to these moduli, in any context, as vector bundle moduli.

We can, in fact, accomplish much more than the enumeration of the vector bundle moduli. As discussed in [32], in heterotic M-theory a bulk space $M5$-brane, wrapped on a holomorphic curve in the Calabi-Yau threefold, can collide with an "end-of-the-world" boundary brane. This collision, called a "small instanton" phase transition, modifies the smooth vector bundle V on the boundary brane, first to a singular "torsion free sheaf" (the small instanton) and then to another smooth vector bundle V', generically of a different topological type than V. During this process, the number of vector bundle moduli changes. In this paper, we will compute, within the class of chirality altering small instanton transitions, the specific change in the number of vector bundle moduli. The new moduli that appear during the collision are termed "transition moduli", and we will elucidate their origin and structure in detail. These transition moduli, in interaction with the M-brane moduli, determine much of the dynamics of small instanton phase transitions. For example, their vacuum values control the strength of non-perturbative corrections to the superpotential. In a cosmological context, the transition moduli will be responsible for processes such as reheating, the cosmological constant and the "bounce" dynamics in Ekpyrotic theories of the early universe.

Specifically, in this paper we will do the following. In section 2, we discuss the explicit relationship between the instanton moduli of an exact solution of the hermitean Yang-Mills equations and the vector bundle moduli of the associated stable, holomorphic vector bundle. In the simple context of compactification on a complex, one-dimensional torus, we compute the number of moduli and their moduli space both analytically, in the hermitean Yang-Mills approach, and algebraically, from the vector bundle. We show that we get the same answer either way. Unfortunately, this is essentially the only case where the analytic moduli can be completely described. This motivates our approach in the rest of the paper, where we will enumerate and describe the properties of these moduli within the context of stable, holomorphic vector bundles. In section 3, we present a brief review of the theory of irreducible, stable, holomorphic $\mathrm{SU}(n)$ vector bundles over elliptically fibered Calabi-Yau threefolds. We also define the concept of a positive spectral cover, and give the criteria for the positivity of spectral covers over threefolds with base $B = \mathbb{F}_r$. An explict discussion of the origin and the computation of the number of vector bundle moduli in this context is given in section 4. Section 5 is devoted to moduli in chirality changing small instanton phase transitions. First, we briefly discuss the basic concepts of these transitions and then explicitly compute the number of transition moduli. We then show that these moduli can be localized on a surface within the Calabi-Yau threefold. This surface is the "lift" of the $M5$-brane curve. Specifically, we identify the transition moduli with the deformations of the restriction of the spectral cover to this surface. In section 6, we give a second interpretation of the transition moduli by evaluating them on the $M5$-brane curve itself. In this context, they appear as the holomorphic sections of a certain rank n vector bundle on the $M5$-brane curve. We calculate this vector bundle explicitly.

This is the first in a series of papers that will discuss the properties of vector bundle moduli. The present paper is, necessarily, somewhat mathematical in its content. This reflects that fact that one must work directly with holomorphic vector bundles, since solutions of the associated hermitean Yang-Mills equations are unknown. Be that as it may, the theory of vector bundle moduli potentially has considerable physical importance. We will elucidate the more physical aspects of these moduli in future publications.

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2. Vector bundle moduli on a torus

Consider an $N = 1$ supersymmetric SU(m) Yang-Mills theory in flat six-dimensional space. The action is given by

$$S_{YM} = -\frac{1}{4g_6^2}\int \mathrm{d}^6x \,\mathrm{tr}\, F_{MN}^2 + \cdots\,, \tag{2.1}$$

where $M, N = 0, 1, \ldots, 5$, g_6 is the six-dimensional coupling parameter and the dots stand for terms containing fermions. Now, compactify this theory on a torus, T^2. To do this, begin by splitting the coordinates x^M into x^μ, $\mu = 0, 1, \ldots, 3$ parameterizing four-dimensional Minkowski space, $\mathbb{R}^4$, and complex coordinates z,$\bar{z}$ on T^2. Similarly, the SU(m) Lie algebra valued connection A_M decomposes into the four-dimensional gauge connection A_μ on $\mathbb{R}^4$ and two scalars A_z and $A_{\bar{z}}(= A_z^\dagger)$. We now want to search for solutions of the Yang-Mills equations restricted to T^2 that are Lie algebra valued in a subgroup SU(n) $\subset$ SU(m) and preserve $N = 1$ supersymmetry on $\mathbb{R}^4$. For this to be the case, A_z and $A_{\bar{z}}$ must satisfy the hermitean Yang-Mills equations which, on T^2, are simply the zero curvature equation

$$F_{z\bar{z}} = 0\,, \tag{2.2}$$

where

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]\,. \tag{2.3}$$

That is, A_z is a flat connection Lie algebra valued in SU(n). If we express all quantities in terms of the real coordinates y_1,y_2 on T^2, then the connection on the torus is written as A_i for $i = 1, 2$. The most general solution of (2.2) is well-known and given by the Wilson line

$$A_i = \mathrm{diag}(A_i^1, \ldots, A_i^n)\,, \qquad \sum_{k=1}^{n} A_i^k = 0\,, \tag{2.4}$$

where

$$A_i^k = C_i^k + \frac{n_i^k}{R_i}\,, \tag{2.5}$$

R_1,R_2 are the radii of the torus T^2 and $n_1^k = l_k$, $n_2^k = m_k$ where l_k, m_k for $k = 1, \ldots, n$ are any integers satisfying

$$\sum_{k=1}^{n} l_k = 0\,, \qquad \sum_{k=1}^{n} m_k = 0\,. \tag{2.6}$$

The parameters C_i^k for $i = 1, 2$ are real numbers subject to the constraints

$$\sum_{k=1}^{n} C_i^k = 0 \tag{2.7}$$

Hence, only $2(n-1)$ of the parameters C_i^k are independent. We conclude, therefore, that the zero curvature equation (2.2) has constant solutions (2.4), (2.5) on T^2 that are not gauge equivalent to zero. They are, however, periodic with periods R_1 and R_2 in the two compact torus directions. These $2(n-1)$ real constants are the "instanton moduli" of this gauge configuration. It follows that the moduli space of flat, SU(n) valued Yang-Mills gauge connections has real dimension $2(n-1)$ and is given by

$$\mathcal{M}_{\text{flat}}(\mathrm{SU}(n)) \simeq \frac{(T^2)^{n-1}}{\Sigma_n}, \tag{2.8}$$

where Σ_n denotes the group of permutations of A_i^k. This space can be shown to be equivalent to the $n-1$ dimensional complex moduli space

$$\mathcal{M}_{\text{flat}}(\mathrm{SU}(n)) \simeq \mathbb{CP}^{n-1}. \tag{2.9}$$

Having found the solution (2.4) of the zero curvature equation for SU(n) Lie algebra valued gauge connections on a simple torus, we see that in the compactification proceedure these moduli will appear as functions, $A_i^k(x^\mu)$, of the coordinates x^μ, $\mu = 0, 1, \ldots, 3$ on $\mathbb{R}^4$. The solution (2.4) is interpreted as the vacuum expectation values

$$C_i^k = < A_i^k(x^\mu) > . \tag{2.10}$$

These expectation values break the gauge group SU(m) down to U$(m-n)$, which is the commutant of SU(n) in SU(m). The fluctuations

$$\phi_i^k(x^\mu) = A_i^k(x^\mu) - C_i^k \tag{2.11}$$

will appear as massless scalar fields in the four-dimensional effective action and are singlets under U$(m-n)$. Returning to complex coordinates z,$\bar{z}$ on T^2, these scalars

$$\phi_z^k = \sqrt{2}\left(\phi_1^k + i\phi_2^k\right), \qquad \sum_{k=1}^{n} \phi_z^k = 0 \tag{2.12}$$

form the lowest components of $n-1$, $N=1$ chiral supermultiplets. The bosonic part of the four-dimensional $N=1$ supersymmentric effective action takes the form

$$S_{YM} = -\frac{1}{4g_4^2}\int \mathrm{d}^4x \left(\mathrm{tr}\, F_{\mu\nu}^2 + 4\sum_{k=1}^{n} \partial_\mu \phi_{\bar{z}}^k \partial^\mu \phi_z^k \cdots\right), \tag{2.13}$$

where g_4 is the four-dimensional coupling parameter, $F_{\mu\nu}$ is the bosonic field strength of the U$(m-n)$ Lie algebra valued super Yang-Mills connection and $\phi_{\bar{z}}^k = \phi_z^{k*}$. There are, of course, also chiral supermultiplets charged under U$(m-n)$ indicated in the action (2.13) by the dots. We conclude that, for the simple case of the torus T^2, one can completely solve the hermitean Yang-Mills equations and determine the instanton moduli. Be that as it may, it is of interest to see whether one could arrive at the same information by analyzing the associated holomorphic vector bundle. We turn to this now.

It is well-known that every stable U(n) holomorphic vector bundle V on T^2 is a sum of n line bundles, which we denote by L_i, each with vanishing first Chern class. That is

$$V = \oplus_{i=1}^{n} L_i \qquad c_1(L_i) = 0\,. \tag{2.14}$$

The additional restriction to structure group SU(n) implies that the product of all the line bundles L_i is the trivial bundle

$$\otimes_{i=1}^{n} L_i = \mathcal{O}\,. \tag{2.15}$$

On the torus, the fact that a line bundle has vanishing first Chern class implies that it has a meromorphic section with exactly one zero and one pole. Denote the zero of the meromorphic section of line bundle L_i by Q_i. The pole can be chosen to be the same point for all line bundles L_i and will be denoted by P. The divisor $Q_i - P$ has zero degree and, therefore, the corresponding line bundle $\mathcal{O}(Q_i) \otimes \mathcal{O}(P)^{-1}$ has vanishing first Chern class. Clearly

$$L_i \simeq \mathcal{O}(Q_i) \otimes \mathcal{O}(P)^{-1}\,. \tag{2.16}$$

Therefore, line bundle L_i is uniquely determined by the point Q_i and the deformation of this bundle specified by how this point can move in the torus. That is, the moduli space of the line bundle L_i is isomorphic to the torus itself. Note that using (2.16), the SU(n) condition (2.15) becomes

$$\sum_{i=1}^{n} (Q_i - P) = 0\,. \tag{2.17}$$

Thus, every stable SU(n) holomorphic vector bundle V is determined by specifying n points Q_i in T^2 up to permutations, only $n-1$ of them being independent due to condition (2.17). Therefore, the moduli space of a stable SU(n) holomorphic vector bundle is given by

$$\mathcal{M}_{\text{bundle}}(\mathrm{SU}(n)) \simeq \frac{(T^2)^{n-1}}{\Sigma_n} \simeq \mathbb{CP}^{n-1}\,. \tag{2.18}$$

Comparing equations (2.8) and (2.18) we see that the two moduli spaces are identical. We conclude that, in a straightforward manner, the number of complex vector bundle moduli, $n-1$, and the moduli space, $\mathbb{CP}^{n-1}$, can be determined for a stable SU(n) holomorphic vector bundle on T^2. Furthermore, both the number of vector bundle moduli and their moduli space exactly correspond to the instanton moduli determined above by explicitly solving the hermitean Yang-Mills equations.

Clearly the structure of moduli can be determined in two ways, either by solving the hermitean Yang-Mills equations or by constructing the associated holomorphic vector bundle and finding its deformations. In the simple case of the torus T^2, either method was efficacious. However, let us consider complex manifolds X where $\dim_{\mathbb{C}} X \geq 2$. The hermitean Yang-Mills equations now become

$$F_{ab} = F_{\bar{a}\bar{b}} = 0\,, \qquad g^{a\bar{b}} F_{a\bar{b}} = 0\,. \tag{2.19}$$

For flat manifolds X of complex dimension two, the solutions of the hermitean Yang-Mills equations can be constructed using the methods of ADHM [33]. However, for non-flat twofolds and for all complex manifolds X with $\dim_{\mathbb{C}} X \geq 3$, there are no known solutions of these equations. Hence, the number and structure of moduli cannot be determined in this way. However, the associated stable holomorphic vector bundles can be constructed, at least for elliptically fibered Calabi-Yau manifolds using the methods of [8, 9] and extended in [13]. Is it possible, therefore, that the number and structure of moduli can be determined by studying the deformations of these holomorphic vector bundles? The answer is affirmative, as we will show in detail in the remainder of this paper for stable SU(n) holomorphic bundles over elliptically fibered Calabi-Yau threefolds.

3. Review of spectral covers and SU(n) vector bundles

3.1 Elliptically fibered Calabi-Yau threefolds

In this paper, we will consider Calabi-Yau threefolds, X, that are structured as elliptic curves fibered over a base surface, B. We denote the natural projection as $\pi : X \to B$ and by $\sigma : B \to X$ the analytic map that defines the zero section.

A simple representation of an elliptic curve is given in the projective space $\mathbb{CP}^2$ by the Weierstrass equation

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3 \,, \tag{3.1}$$

where (x, y, z) are the homogeneous coordinates of $\mathbb{CP}^2$ and g_2, g_3 are constants. This same equation can represent the elliptic fibration, X, if the coefficients g_2, g_3 in the Weierstrass equation are functions over the base surface, B. The correct way to express this globally is to replace the projective plane $\mathbb{CP}^2$ by a $\mathbb{CP}^2$-bundle $P \to B$ and then require that g_2, g_3 be sections of appropriate line bundles over the base. If we denote the conormal bundle to the zero section $\sigma(B)$ by $\mathcal{L}$, then $P = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$, where $\mathbb{P}(W)$ stands for the projectivization of a vector bundle W. There is a hyperplane line bundle $\mathcal{O}_P(1)$ on P which corresponds to the divisor $\mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^3) \subset P$ and the coordinates x, y, z are sections of $\mathcal{O}_P(1) \otimes \mathcal{L}^2, \mathcal{O}_P(1) \otimes \mathcal{L}^3$ and $\mathcal{O}_P(1)$ respectively. It then follows from (3.1) that the coefficients g_2 and g_3 are sections of $\mathcal{L}^4$ and $\mathcal{L}^6$.

It is useful to define new coordinates, $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, on X by $x = \mathbf{XZ}$, $y = \mathbf{Y}$ and $z = \mathbf{Z}^3$. It follows that $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are now sections of line bundles

$$\mathbf{X} \sim \mathcal{O}_X(2\sigma) \otimes \mathcal{L}^2 \,, \qquad \mathbf{Y} \sim \mathcal{O}_X(3\sigma) \otimes \mathcal{L}^3 \,, \qquad \mathbf{Z} \sim \mathcal{O}_X(\sigma) \tag{3.2}$$

respectively, where we use the fact that $\mathcal{O}_X(3\sigma) = \mathcal{O}_P(1)$. The coefficients g_2 and g_3 remain sections of line bundles

$$g_2 \sim \mathcal{L}^4 \,, \qquad g_3 \sim \mathcal{L}^6 \,. \tag{3.3}$$

The symbol "$\sim$" simply means "section of".

The requirement that elliptically fibered threefold, X, be a Calabi-Yau space constrains the first Chern class of the tangent bundle, TX, to vanish. That is,

$$c_1(TX) = 0 \,. \tag{3.4}$$

It follows from this that

$$\mathcal{L} = K_B^{-1} \,, \tag{3.5}$$

where K_B is the canonical bundle on the base, B. Condition (3.5) is rather strong and restricts the allowed base spaces of an elliptically fibered Calabi-Yau threefold to be del Pezzo, Hirzebruch and Enriques surfaces, as well as certain blow-ups of Hirzebruch surfaces [34].

3.2 Spectral cover description of SU(n) vector bundles

As discussed in [8, 9], SU(n) vector bundles over an elliptically fibered Calabi-Yau threefold can be explicitly constructed from two mathematical objects, a divisor $\mathcal{C}$ of X, called the spectral cover, and a line bundle $\mathcal{N}$ on $\mathcal{C}$. In this paper, we will consider only stable SU(n) vector bundles constructed from irreducible spectral covers. The moduli of semi-stable vector bundles associated with reducible spectral covers will be discussed elsewhere. In addition, we will impose the condition that the spectral cover be a "positive" divisor.

3.3 Spectral cover

A spectral cover, $\mathcal{C}$, is a surface in X that is an n-fold cover of the base B. That is, $\pi_{\mathcal{C}} : \mathcal{C} \to B$. The general form for a spectral cover is given by

$$\mathcal{C} = n\sigma + \pi^*\eta\,, \tag{3.6}$$

where σ is the zero section and η is some curve in the base B. The terms in (3.6) can be considered either as elements of the homology group $H_4(X, \mathbb{Z})$ or, by Poincare duality, as elements of cohomology $H^2(X, \mathbb{Z})$.

In terms of the coordinates $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ introduced above, it can be shown that the spectral cover can be represented as the zero set of the polynomial

$$s = a_0\mathbf{Z}^n + a_2\mathbf{X}\mathbf{Z}^{n-2} + a_3\mathbf{Y}\mathbf{Z}^{n-3} + \cdots + a_n\mathbf{X}^{\frac{n}{2}} \tag{3.7}$$

for n even and ending in $a_n\mathbf{X}^{\frac{n-3}{2}}\mathbf{Y}$ if n is odd, along with the relation s (3.2). This tells us that the polynomial s must be a holomorphic section of the line bundle of the spectral cover, $\mathcal{O}_X(\mathcal{C})$. That is,

$$s \sim \mathcal{O}_X(n\sigma + \pi^*\eta)\,. \tag{3.8}$$

It follows from this and equations (3.2) and (3.3), that the coefficients a_i in the polynomial s must be sections of the line bundles

$$a_i \sim \pi^* K_B^i \otimes \mathcal{O}_X(\pi^*\eta) \tag{3.9}$$

for $i = 1, \ldots, n$ where we have used expression (3.5).

In order to describe vector bundles most simply, there are two properties that we require the spectral cover to possess. The first, which is shared by all spectral covers, is that

- $\mathcal{C}$ must be an effective class in $H_4(X, \mathbb{Z})$.

This property is simply an expression of the fact the spectral cover must be an actual surface in X. It can easily be shown that

$$\mathcal{C} \subset X \text{ is effective } \iff \eta \text{ is an effective class in } H_2(B, \mathbb{Z})\,. \tag{3.10}$$

The second property that we require for the spectral cover is that

- $\mathcal{C}$ is an irreducible surface.

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This condition is imposed because it guarantees that the associated vector bundle is stable. Deriving the conditions under which $\mathcal{C}$ is irreducible is not completely trivial and will be discussed elsewhere. Here, we will simply state the results. First, recall from (3.9) that $a_i \sim \pi^* K_B^i \otimes \mathcal{O}_X(\pi^*\eta)$ and, hence, the zero locus of a_i is a divisor, $D(a_i)$, in X. Then, we can show that $\mathcal{C}$ is an irreducible surface if

$$D(a_0) \text{ is an irreducible divisor in } X \tag{3.11}$$

and

$$D(a_n) \text{ is an effective class in } H_4(X, \mathbb{Z}) \, . \tag{3.12}$$

Using Bertini's theorem, it can be shown that condition (3.11) is satisfied if the linear system $|\eta|$ is base point free. "Base point free" means that for any $b \in B$, we can find a member of the linear system $|\eta|$ that does not pass through the point b.

In order to make these concepts more concrete, we take, as an example, the base surface to be

$$B = \mathbb{F}_r \tag{3.13}$$

and derive the conditions under which (3.10), (3.12) and (3.11) are satisfied. Recall [] that the homology group $H_2(\mathbb{F}_r, \mathbb{Z})$ has as a basis the effective classes $\mathcal{S}$ and $\mathcal{E}$ with intersection numbers

$$\mathcal{S}^2 = -r \, , \qquad \mathcal{S} \cdot \mathcal{E} = 1 \, , \qquad \mathcal{E}^2 = 0 \, . \tag{3.14}$$

Then, in general, $\mathcal{C}$ is given by expression (3.6) where

$$\eta = a\mathcal{S} + b\mathcal{E} \tag{3.15}$$

and a,b are integers. One can easily check that η is an effective class in $\mathbb{F}_r$, and, hence, that $\mathcal{C}$ is an effective class in X, if and only if

$$a \geq 0 \, , \qquad b \geq 0 \, . \tag{3.16}$$

It is also not too hard to demonstrate that the linear system $|\eta|$ is base point free if and only if

$$b \geq ar \, . \tag{3.17}$$

Imposing this constraint then implies that $D(a_0)$ is an irreducible divisor in X. Finally, we can show that for $D(a_n)$ to be effective in X one must have

$$a \geq 2n \, , \qquad b \geq n(r+2) \, . \tag{3.18}$$

Combining conditions (3.17) and (3.18) then guarantees that $\mathcal{C}$ is an irreducible surface. We now turn to the second mathematical object that is required to specify an SU(n) vector bundle.

In this paper, we require a third condition on the spectral cover that

- $\mathcal{C}$ is positive.

This condition is imposed because, as we will see below, it allows one to use the Kodaira Vanishing theorem and the Lefschetz theorem when evaluating the number of vector bundle moduli. These theorems greatly simplify this calculation. By definition, $\mathcal{C}$ is positive (or ample) if and only if the first Chern class of the associated line bundle $\mathcal{O}_X(\mathcal{C})$ is a positive class in $H^2_{DR}(X,\mathbb{R})$. An equivalent, and from the point of view of this paper more useful, definition of positive is the following. $\mathcal{C}$ is a positive divisor if and only if

$$\mathcal{C}\cdot c > 0 \tag{3.19}$$

for every effective curve c in X.

In order to make this concept more concrete, we take, as an example, $B=\mathbb{F}_r$ and $\eta = a\mathcal{S}+b\mathcal{E}$, where a,b are integers. To proceed, we must identify all of the effective curves c in X. These are easily shown to be

$$(\pi^*\mathcal{S})\cdot\sigma\,,\qquad(\pi^*\mathcal{E})\cdot\sigma\,,\qquad F\,, \tag{3.20}$$

where F is the class of a generic fiber of X. First consider $\mathcal{C}\cdot F$. Using

$$\sigma\cdot F=1\,,\qquad(\pi^*\mathcal{S})\cdot F=(\pi^*\mathcal{E})\cdot F=0\,, \tag{3.21}$$

it is straightforward to show that

$$\mathcal{C}\cdot F=n\,, \tag{3.22}$$

which is positive. The fact that $\sigma\cdot F=1$ is obvious. The relation $(\pi^*\mathcal{S})\cdot F=(\pi^*\mathcal{E})\cdot F=0$ follows from the fact that one can always choose a representation of the class F that does not intersect either $\pi^*\mathcal{S}$ or $\pi^*\mathcal{E}$. To evaluate $\mathcal{C}\cdot((\pi^*\mathcal{S})\cdot\sigma)$ and $\mathcal{C}\cdot((\pi^*\mathcal{E})\cdot\sigma)$ requires the somewhat more subtle fact that

$$\sigma\cdot\sigma=-(\pi^*c_1(B))\cdot\sigma\,, \tag{3.23}$$

where $c_1(B)$ is the first Chern class of the base B. This can be proven as follows. First, note that by adjunction

$$K_\sigma=(K_X\otimes\mathcal{O}_X(\sigma))|_\sigma\,, \tag{3.24}$$

where K_X and K_σ are the canonical line bundles on X and $\sigma\subset X$ respectively. Since X is a Calabi-Yau threefold, it follows from (3.4) that

$$c_1(K_X)=-c_1(TX)=0 \tag{3.25}$$

and, hence, that K_X is the trivial bundle

$$K_X=\mathcal{O}\,. \tag{3.26}$$

Therefore, equation (3.24) becomes

$$K_\sigma=\mathcal{O}_X(\sigma)|_\sigma\,. \tag{3.27}$$

Pulling this back onto the base B using the map $\sigma: B\to X$ yields

$$K_B=\sigma^*\mathcal{O}_X(\sigma)|_\sigma\,, \tag{3.28}$$

where K_B is the canonical bundle of B. Therefore

$$c_1(K_B) = c_1(\sigma^* \mathcal{O}_X(\sigma)|_\sigma) \,. \tag{3.29}$$

Using the relation

$$c_1(K_B) = -c_1(B) \,, \tag{3.30}$$

it follows that

$$-c_1(B) = c_1(\sigma^* \mathcal{O}_X(\sigma)|_\sigma) \,. \tag{3.31}$$

Pulling this expression back onto σ using the map $\pi : X \to B$, one finds

$$-(\pi^* c_1(B)) \cdot \sigma = (\pi^* c_1(\sigma^* \mathcal{O}_X(\sigma)|_\sigma)) \cdot \sigma \,. \tag{3.32}$$

Note, however, that $c_1(\sigma^* \mathcal{O}_X(\sigma)|_\sigma)$ is a curve on B constructed by pulling back the line bundle associated with σ restricted to itself. Therefore $(\pi^* c_1(\sigma^* \mathcal{O}_X(\sigma)|_\sigma)) \cdot \sigma$ is precisely the curve $\sigma \cdot \sigma$ in X, thus establishing relation (3.23).

Using (3.21), (3.23) and the relation

$$(\pi^* \gamma_1) \cdot (\pi^* \gamma_2) = \pi^* (\gamma_1 \cdot \gamma_2) \tag{3.33}$$

for any two curves γ_1, γ_2 on B, one can now evaluate $\mathcal{C} \cdot ((\pi^* \mathcal{S}) \cdot \sigma)$ and $\mathcal{C} \cdot ((\pi^* \mathcal{E}) \cdot \sigma)$. To do this, one should note that for

$$B = \mathbb{F}_r \,, \tag{3.34}$$

the Chern classes are known and given by

$$c_1(\mathbb{F}_r) = 2\mathcal{S} + (r+2)\mathcal{E} \,, \qquad c_2(\mathbb{F}_r) = 4 \,. \tag{3.35}$$

Combining everything, we find that

$$\mathcal{C} \cdot ((\pi^* \mathcal{S}) \cdot \sigma) = n(r-2) - ar + b \,, \tag{3.36}$$

which will be positive if and only if

$$b > ar - n(r-2) \,. \tag{3.37}$$

Similarly

$$\mathcal{C} \cdot ((\pi^* \mathcal{E}) \cdot \sigma) = -2n + a \,. \tag{3.38}$$

For this to be positive one must have

$$a > 2n \,. \tag{3.39}$$

We conclude that the spectral cover $\mathcal{C}$ is positive if and only if conditions (3.37) and (3.39) are satisfied.

3.4 The line bundle $\mathcal{N}$

As discussed in [8, 9], in addition to the spectral cover it is necessary to specify a line bundle, $\mathcal{N}$, over $\mathcal{C}$. For SU(n) vector bundles, this line bundle must be a restriction of a global line bundle on X (which we will again denote by $\mathcal{N}$), satisfying the condition

$$c_1(\mathcal{N}) = n\left(\frac{1}{2}+\lambda\right)\sigma + \left(\frac{1}{2}-\lambda\right)\pi^*\eta + \left(\frac{1}{2}+n\lambda\right)\pi^*c_1(B)\,, \tag{3.40}$$

where $c_1(\mathcal{N})$, $c_1(B)$ are the first Chern classes of $\mathcal{N}$ and B respectively and λ is, a priori, a rational number. Since $c_1(\mathcal{N})$ must be an integer class, it follows that either

$$n \quad \text{is odd}\,, \qquad \lambda = m + \frac{1}{2} \tag{3.41}$$

or

$$n \quad \text{is even}\,, \qquad \lambda = m\,, \qquad \eta = c_1(B) \quad \text{mod } 2\,, \tag{3.42}$$

where $m \in \mathbb{Z}$.

3.5 SU(n) vector bundle

Given a spectral cover, $\mathcal{C}$, and a line bundle, $\mathcal{N}$, satisfying the above properties, one can now uniquely construct an SU(n) vector bundle, V. This can be accomplished in two ways. First, as discussed in [9], the vector bundle can be directly constructed using the associated Poincare bundle, $\mathcal{P}$. The result is that

$$V = \pi_{1*}(\pi_2^*\mathcal{N}\otimes\mathcal{P})\,, \tag{3.43}$$

where π_1 and π_2 are the two projections of the fiber product $X \times_B \mathcal{C}$ onto the two factors X and $\mathcal{C}$. We refer the reader to [9, 13] for a detailed discussion. Equivalently, V can be constructed directly from $\mathcal{C}$ and $\mathcal{N}$ using the Fourier-Mukai transformation, as discussed in [8, 9]. Both of these constructions work in reverse, yielding the spectral data $(\mathcal{C}, \mathcal{N})$ up to the overall factor of K_B given the vector bundle V. Throughout this paper we will indicate this relationship between the spectral data and the vector bundle by writing

$$(\mathcal{C}, \mathcal{N}) \longleftrightarrow V. \tag{3.44}$$

The Chern classes for the SU(n) vector bundle V have been computed in [8] and [13, 15]. The results are

$$c_1(V) = 0 \tag{3.45}$$

since $\operatorname{tr} F = 0$ for the structure group SU(n),

$$c_2(V) = \pi^*\eta\cdot\sigma - \frac{1}{24}\pi^*c_1(B)^2(n^3-n) + \frac{1}{2}\left(\lambda^2-\frac{1}{4}\right)n\pi^*(\eta\cdot(\eta-nc_1(B))) \tag{3.46}$$

and

$$c_3(V) = 2\lambda\sigma\cdot\pi^*(\eta\cdot(\eta-nc_1(B)))\,. \tag{3.47}$$

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Finally, we note that it was shown in [13] that

$$N_{\text{gen}} = \frac{c_3(V)}{2} \tag{3.48}$$

gives the number of quark and lepton generations.

To conclude, in this section we have discussed the construction and properties of stable SU(n) vector bundles associated with positive, irreducible spectral covers over elliptically fibered Calabi-Yau threefolds. For the remainder of this paper, for brevity, we will refer to such bundles simply as "stable SU(n) vector bundles". In order to make these concepts more transparent, we now present several examples.

Example 1. Consider a vector bundle specified by $B = \mathbb{F}_1$, $G = SU(3)$ and spectral cover

$$\mathcal{C} = 3\sigma + \pi^* \eta\,, \tag{3.49}$$

where

$$\eta = 7\mathcal{S} + 12\mathcal{E}\,. \tag{3.50}$$

That is, $r = 1$, $n = 3$, $a = 7$, and $b = 12$. Clearly, these parameters satisfy (3.16) as well as (3.17), (3.18) and, hence, $\mathcal{C}$ is effective and irreducible respectively. Furthermore, (3.37) and (3.39) are satisfied, implying that $\mathcal{C}$ is a positive divisor. Now choose the line bundle $\mathcal{N}$ with

$$\lambda = \frac{1}{2}\,, \tag{3.51}$$

which satisfies condition (3.41) for $m = 0$. Using (3.14), (3.35), (3.49) and (3.50) we find from (3.47) and (3.48) that

$$N_{\text{gen}} = 13\,. \tag{3.52}$$

is the number of generations.

Example 2. As a second example, we again choose $B = \mathbb{F}_1$ and $G = SU(3)$, but now consider spectral cover

$$\mathcal{C}' = 3\sigma + \pi^* \eta'\,, \tag{3.53}$$

where

$$\eta' = 7\mathcal{S} + 13\mathcal{E}\,. \tag{3.54}$$

That is, $r = 1$, $n = 3$, $a' = 7$, and $b' = 13$, which clearly satisfy (3.16), (3.17), (3.18) and (3.37), (3.39). Hence $\mathcal{C}'$ is an effective, irreducible,positive divisor. We again take

$$\lambda = \frac{1}{2}\,. \tag{3.55}$$

Using (3.14), (3.35), (3.53) and (3.54) we find from (3.47) and (3.48) that

$$N_{\text{gen}} = 17\,. \tag{3.56}$$

is the number of generations.

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Example 3. As a final example, we again take $B = \mathbb{F}_1$ and $G = \mathrm{SU}(3)$, but consider

$$\mathcal{C}' = 3\sigma + \pi^* \eta' \,, \tag{3.57}$$

where

$$\eta' = 8\mathcal{S} + 12\mathcal{E} \,. \tag{3.58}$$

That is, $r = 1$, $n = 3$, $a' = 8$, and $b' = 12$, which clearly satisfy (3.16), (3.17), (3.18) and (3.37), (3.39). Hence $\mathcal{C}'$ is an effective, irreducible, positive divisor. Again, let

$$\lambda = \frac{1}{2} \,. \tag{3.59}$$

Using (3.14), (3.35), (3.57) and (3.58) we find from (3.47) and (3.48) that

$$N_{\mathrm{gen}} = 16 \,. \tag{3.60}$$

is the number of generations.

We will refer to these examples elsewhere in this paper to illustrate our results on vector bundle moduli.

4. The moduli of stable $\mathrm{SU}(n)$ vector bundles

The moduli of holomorphic vector bundles have been discussed, in a generic way, in [8, 9] and elsewhere. For example, general properties of heterotic vector bundle moduli on certain elliptically fibered Calabi-Yau spaces, as well as on the dual F-theory compactifications, have been discussed in [35]–[41]. In this section, we present an explicit calculation of the number of moduli of stable, irreducible $\mathrm{SU}(n)$ holomorphic vector bundles over elliptically fibered Calabi-Yau threefolds. For specificity, we present an explicit formula for the number of vector bundle moduli when the Calabi-Yau threefold has base $B = \mathbb{F}_r$.

It is clear from expression (3.44) that the number of moduli of a stable $\mathrm{SU}(n)$ vector bundle V is determined by the number of parameters specifying its spectral cover $\mathcal{C}$ and by the size of the space of holomorphic line bundles $\mathcal{N}$ defined on $\mathcal{C}$. We begin by considering the spectral cover.

4.1 Parameters of $\mathcal{C}$

The spectral cover $\mathcal{C}$ is a divisor of the Calabi-Yau threefold X and, hence, uniquely determines a line bundle $\mathcal{O}_X(C)$ on X. Then $H^0(X, \mathcal{O}_X(C))$ is the space of holomorphic sections of $\mathcal{O}_X(C)$. We denote its dimension by $h^0(X, \mathcal{O}_X(C))$. It follows that there must exist h^0 holomorphic sections $s_1, \ldots, s_{h^0}$ that span this space. Note that the zero locus of each holomorphic section of the form

$$s_{\{a_i\}} = \Sigma_{i=1}^{h^0} a_i s_i \,, \tag{4.1}$$

for fixed complex coefficients a_i, determines an effective divisor $\mathcal{C}_{\{a_i\}}$ of X. Running over all a_i gives the set, $|\mathcal{C}|$, of effective divisors associated with $\mathcal{O}_X(C)$. Clearly

$$|\mathcal{C}| = \mathbb{P}H^0(X, \mathcal{O}_X(C)) \,, \tag{4.2}$$

where the right hand side is the projectivization of $H^0(X, \mathcal{O}_X(C))$. It follows that

$$\dim |\mathcal{C}| = h^0(X, \mathcal{O}_X(\mathcal{C})) - 1\,. \tag{4.3}$$

This quantity counts the number of parameters specifying the spectral cover $\mathcal{C}$. We now consider the line bundles $\mathcal{N}$ over $\mathcal{C}$.

4.2 The space of line bundles $\mathcal{N}$

The set of holomorphic line bundles $\mathcal{N}$ over the spectral cover $\mathcal{C}$ is, by definition, determined by the set of holomorphic transition functions allowed on $\mathcal{C}$. These, in turn, are specified as the closed but not exact elements of the multiplicative group $C^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^*)$ of non-vanishing holomorphic functions on the intersection of any two open sets in the atlas of $\mathcal{C}$. That is, the group of line bundles of $\mathcal{C}$ is given by

$$\mathrm{Pic}(\mathcal{C}) = H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^*)\,, \tag{4.4}$$

where $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^*)$ is the first Čech cohomology group of $\mathcal{O}_{\mathcal{C}}^*$ on $\mathcal{C}$. Clearly then, the size of the space of line bundles $\mathcal{N}$ over $\mathcal{C}$ is specified by

$$\dim \mathrm{Pic}(\mathcal{C}) = h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^*)\,. \tag{4.5}$$

Putting the results of the last two subsections together, we see that the number of moduli of a stable SU(n) vector bundle V, which we will denote by $n(V)$, is given by

$$n(V) = (h^0(X, \mathcal{O}_X(\mathcal{C})) - 1) + h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^*)\,. \tag{4.6}$$

We now turn to the explicit evaluation of each of the terms in this expression.

4.3 The evaluation of $h^0(X, \mathcal{O}_X(\mathcal{C}))$

Our basic approach to evaluating $h^0(X, \mathcal{O}_X(\mathcal{C}))$ is through the Riemann-Roch theorem which, in this context, states that

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = h^0(X, \mathcal{O}_X(\mathcal{C})) - h^1 + h^2 - h^3\,, \tag{4.7}$$

where $\chi_E(X, \mathcal{O}_X(\mathcal{C}))$ is the Euler characteristic and h^q for $q = 1, 2, 3$ are the dimensions of the higher cohomology groups of X evaluated in $\mathcal{O}_X(\mathcal{C})$. To proceed, we will use the fact that the stable SU(n) vector bundles that we consider in this paper are positive. This allows us to employ the Kodaira vanishing theorem which states that, for positive line bundle $\mathcal{O}_X(\mathcal{C})$ and $\dim_{\mathbb{C}} X = m$,

$$H^q(X, \Omega_X^p(\mathcal{O}_X(\mathcal{C})) = 0 \tag{4.8}$$

for $p + q > m$. In our case $m = 3$ and consider $p = 3$. Then

$$\Omega_X^3(\mathcal{O}_X(\mathcal{C})) = \Omega_X^3(TX) \otimes \mathcal{O}_X(\mathcal{C})\,. \tag{4.9}$$

However, since we are on a threefold

$$\Omega_X^3(TX) = K_X\,, \tag{4.10}$$

where K_X is the canonical bundle of X. But X is a Calabi-Yau manifold and, hence, K_X is the trivial line bundle. Therefore,

$$\Omega_X^3(\mathcal{O}_X(\mathcal{C})) = \mathcal{O}_X(\mathcal{C})\,. \tag{4.11}$$

In this situation, the vanishing theorem then states that

$$H^q(X, \mathcal{O}_X(\mathcal{C})) = 0 \tag{4.12}$$

for $q > 0$. It follows that $h^q = 0$ for $q = 1, 2, 3$ and the Riemann-Roch theorem simplifies to

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = h^0(X, \mathcal{O}_X(\mathcal{C}))\,. \tag{4.13}$$

Therefore, to evaluate $h^0(X, \mathcal{O}_X(\mathcal{C}))$ we need simply to evaluate the Euler characteristic. For the situation at hand, the Euler characteristic is determined from the Atiyah-Singer index theorem to be

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = \int_X ch(\mathcal{O}_X(\mathcal{C})) \wedge Td(TX)\,, \tag{4.14}$$

where ch and Td are the total Chern character and Todd class respectively. This can be evaluated for any elliptically fibered Calabi-Yau threefold of base B. However, throughout this paper, we will do explicit calculations only for the case where

$$B = \mathbb{F}_r\,. \tag{4.15}$$

Recall from the previous section that, in this case, the spectral cover has the generic form

$$\mathcal{C} = n\sigma + \pi^*(a\mathcal{S} + b\mathcal{E})\,, \tag{4.16}$$

where a, b are integers. For the present calculation, we need put no restrictions on a and b. On a Calabi-Yau threefold, (4.14) takes a form

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = \frac{1}{6}\int_X c_1^3(\mathcal{O}_X(\mathcal{C})) + \frac{1}{12}\int_X c_1(\mathcal{O}_X(\mathcal{C})) \wedge c_2(TX)\,. \tag{4.17}$$

The first Chern class $c_1(\mathcal{O}_X(\mathcal{C}))$ is just given by $\mathcal{C}$. The second Chern class of the tangent bundle $c_2(TX)$ has been found in [8] to be

$$c_2(TX) = \pi^*(c_2(B) + 11c_1^2(B)) + 12\sigma \cdot \pi^*(c_1(B))\,. \tag{4.18}$$

Now, by using equation (3.35) for the first and second Chern classes of $\mathbb{F}_r$ and equation (3.23) one finds

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = \frac{n}{3}\left(4n^2 - 1\right) + nab - \left(n^2 - 2\right)(a + b) + ar\left(\frac{n^2}{2} - 1\right) - \frac{n}{2}ra^2\,. \tag{4.19}$$

Of course, there are further restrictions on the integers a and b. These are 1) the non-negativity conditions (3.16) required to make $\mathcal{C}$ effective and 2) the conditions (3.17), (3.18) and (3.37), (3.38) necessary to render $\mathcal{C}$ irreducible and positive. Under these conditions, equation (4.13) is valid and, hence,

$$h^0(X, \mathcal{O}_X(\mathcal{C})) = \frac{n}{3}\left(4n^2-1\right) + nab - \left(n^2-2\right)(a+b) + ar\left(\frac{n^2}{2}-1\right) - \frac{n}{2}ra^2 . \tag{4.20}$$

We now proceed to calculate $h^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})$.

4.4 The evaluation of $h^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})$

In order to evaluate $h^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})$ it will first be necessary to construct some related cohomology groups. The Hurewicz theorem tells us that

$$H_1(X, \mathbb{Z}) = \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} , \tag{4.21}$$

where $[\pi_1(X), \pi_1(X)]$ is the commutator subgroup of $\pi_1(X)$. For most of the Calabi-Yau threefolds of interest to particle physics, the fundamental group will be finite, such as $\pi_1 = \mathbf{1}$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and so on. It follows that

$$H^1_{DR}(X, \mathbb{R}) = 0 . \tag{4.22}$$

However,

$$H^1_{DR}(X, \mathbb{R}) \otimes \mathbb{C} \equiv H^1_{DR}(X, \mathbb{C}) = H^{1,0}_{\bar{\partial}}(X, \mathbb{C}) \oplus H^{0,1}_{\bar{\partial}}(X, \mathbb{C}) . \tag{4.23}$$

We see from this and (4.22) that

$$H^{0,1}_{\bar{\partial}}(X, \mathbb{C}) = 0 . \tag{4.24}$$

Finally, using the Dolbeault theorem

$$H^{0,1}_{\bar{\partial}}(X, \mathbb{C}) \cong H^1(X, \mathcal{O}_X) , \tag{4.25}$$

we conclude that

$$H^1(X, \mathcal{O}_X) = 0 . \tag{4.26}$$

However, the group $H^1(X, \mathcal{O}_X)$ is not quite what we want. What we really need to know is the restriction of this group to the spectral cover $\mathcal{C}$. To find this restriction, we can use the Lefschetz theorem which states the following. For a compact m-fold X and a smooth, positive $m-1$ dimensional submanifold $\mathcal{C} \subset X$, then the map

$$H^p(X, \Omega^q_X) \to H^p(\mathcal{C}, \Omega^q_{\mathcal{C}}) \tag{4.27}$$

induced by the inclusion $i : \mathcal{C} \to X$ is an isomorphism for $p + q \leq m - 2$ and is injective for $p + q = m - 1$. Choosing $p = 1$, $q = 0$ satisfies the isomorphism condition for $m = 3$. Then, using the fact that $\Omega^0 \cong \mathcal{O}$, the Lefschetz theorem becomes

$$H^1(X, \mathcal{O}_X) \cong H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) . \tag{4.28}$$

It follows from this and (4.26) that

$$H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 0\,. \tag{4.29}$$

This is still not quite what we require since we need the first cohomology group evaluated over not $\mathcal{O}_{\mathcal{C}}$ but, rather, $\mathcal{O}^*_{\mathcal{C}}$. This can be obtained as follows. Recall that there is a simple exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}^*_{\mathcal{C}} \to 0\,, \tag{4.30}$$

where the map from $\mathbb{Z} \to \mathcal{O}_{\mathcal{C}}$ is inclusion and $\mathcal{O}_{\mathcal{C}} \to \mathcal{O}^*_{\mathcal{C}}$ is the exponential mapping defined by $\exp(f) = e^{2\pi i f}$. In the usual way, this produces a long exact sequence of cohomology groups given in part by

$$\to H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \to H^2(\mathcal{C}, \mathbb{Z}) \to\,, \tag{4.31}$$

where $H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \to H^2(\mathcal{C}, \mathbb{Z})$ is the Čech coboundary operation δ_1. But, by equation (4.29) $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ vanishes. Hence, the mapping

$$\delta_1 : H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \to H^2(\mathcal{C}, \mathbb{Z}) \tag{4.32}$$

is injective. That is,

$$\mathrm{Pic}(\mathcal{C}) = H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \subset H^2(\mathcal{C}, \mathbb{Z})\,. \tag{4.33}$$

Note that $H^2(\mathcal{C}, \mathbb{Z})$ forms a rigid lattice and, hence, there are no smooth deformations, as there would be in a space over the complex numbers $\mathbb{C}$. We conclude that

$$\dim H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) = 0\,. \tag{4.34}$$

4.5 $n(V)$ for $B = \mathbb{F}_r$

We can now give the final expression for the number of moduli of a positive, stable SU(n) vector bundle over an elliptically fibered Calabi-Yau threefold with base $B = \mathbb{F}_r$. The associated spectral cover is given by

$$\mathcal{C} = n\sigma + \pi^*(a\mathcal{S} + b\mathcal{E})\,, \tag{4.35}$$

where the effectiveness of $\mathcal{C}$ requires

$$a \geq 0\,, \qquad b \geq 0\,, \tag{4.36}$$

the irreducibility of $\mathcal{C}$ demands that

$$b \geq ar\,, \qquad a \geq 2n\,, \qquad b \geq n(r+2) \tag{4.37}$$

and the positivity requirement for $\mathcal{C}$ implies

$$a > 2n\,, \qquad b > ar - n(r-2)\,. \tag{4.38}$$

Using equations (4.6), (4.20) and (4.34), one can conclude the following.

- The number of vector bundle moduli is given by

$$n(V) = \frac{n}{3}\left(4n^2 - 1\right) + nab - \left(n^2 - 2\right)(a+b) + ar\left(\frac{n^2}{2} - 1\right) - \frac{n}{2}ra^2 - 1\,. \tag{4.39}$$

This equation will be essential to the subsequent discussion. To make this result more concrete, we evaluate it for each of the three stable SU(3) vector bundles presented at the end of section 3.

Example 1. In this case $B = \mathbb{F}_1$, $G = \mathrm{SU}(3)$, $a = 7$ and $b = 12$. It follows from (4.39) that the number of vector bundle moduli of V is

$$n(V) = 104\,. \tag{4.40}$$

Example 2. In this case $B = \mathbb{F}_1$, $G = \mathrm{SU}(3)$, $a' = 7$ and $b' = 13$. Then (4.39) implies that

$$n(V') = 118 \tag{4.41}$$

is the number of vector bundle moduli of V'.

Example 3. In this case $B = \mathbb{F}_1$, $G = \mathrm{SU}(3)$, $a' = 8$ and $b' = 12$. Then, it follows from (4.39) that

$$n(V') = 114 \tag{4.42}$$

is the number of vector bundle moduli of V'.

5. Moduli and small instanton transitions

Heterotic M-theory [4, 5] is the compactification of Hořava-Witten theory [3] on a smooth Calabi-Yau threefold with non-vanishing G-flux. At energies below the scale of the Calabi-Yau space, heterotic M-theory appears as a five-dimensional bulk space bounded at either end of the fifth dimension by a four-dimensional end-of-the-world 3-brane. Specifically, these bounding branes are ten-dimensional $S^1/\mathbb{Z}_2$ orbifold fixed planes with six spacelike dimensions wrapped on the Calabi-Yau threefold. As shown by Hořava and Witten [3], anomaly cancellation demands that there must exist an E_8, $N = 1$ supergauge multiplet on each orbifold plane. Compactification allows for the possibility that there is a non-trivial vacuum E_8 gauge configuration on the Calabi-Yau manifold at each end-of-the-world orbifold plane. If this vacuum "instanton" has structure group $G \subset E_8$, then the theory that appears on the low energy 3-brane worldvolume is an $N = 1$ gauge theory with gauge group H, where H is the commutant of G in E_8. In addition, there appear matter supermultiplets with a calculable number of families. Such G-instantons, within the context of elliptically fibered and torus fibered Calabi-Yau threefolds, are discussed in detail in [13, 14, 15].

The vector bundles asociated with the two end-of-the-world instantons are not independent. They are non-trivially correlated by the requirement that the entire theory be anomaly free. As discussed in [13], generally, anomaly freedom is difficult to satisfy for phenomenologically interesting heterotic M-theories with end-of-the-world branes alone. However, anomaly freedom is easily achieved if one allows for $M5$-branes in the bulk space [3, 13, 14]. These branes must be wrapped on an effective holomorphic curve in the Calabi-Yau manifold X to give $N = 1$ supersymmetric bulk 3-branes. If we denote the Poincare dual of this holomorphic curve by the four-form W, then anomaly cancellation tells us that

$$W = c_2(TX) - c_2(V_1) - c_2(V_2)\,, \tag{5.1}$$

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where $c_2(TX)$ is the second Chern class of the tangent bundle of X and $c_2(V_1)$, $c_2(V_2)$ are the second Chern classes of the first and second end-of-the-world vector bundles respectively. Restricting X to be elliptically fibered, as we will henceforth do, W decomposes as

$$W = W_B \cdot \sigma + a_f F \,, \tag{5.2}$$

where $W_B = \pi^* w$ is the lift of a curve w in the base B and a_f is an integer. That is, the holomorphic curve W decomposes into a purely horizontal curve $W_B \cdot \sigma$ and a purely vertical curve proportional to the fiber F. It was shown in [13] that W is an effective class in $H_2(X, \mathbb{Z})$ if and only if w is an effective class in $H(B, \mathbb{Z})$ and $a_f \geq 0$, for any del Pezzo or Enriques base B. This is also true for any Hirzebruch base $B = \mathbb{F}_r$, except when w contains the negative section $\mathcal{S}$ and $r \geq 3$. We will avoid this pathological case in this paper.

Let us now assume that we have a vacuum configuration of heterotic M-theory consisting of two end-of-the-world branes with vector bundles V_1 and V_2 respectively and internal bulk $M5$-branes wrapped on holomorphic curve $W = (\pi^* w) \cdot \sigma + a_f F$. Let the spectral cover describing the SU(n) vector bundle V_1 be given by

$$\mathcal{C} = n\sigma + \pi^* \eta \,, \tag{5.3}$$

where we will always take $\mathcal{C}$ to be effective, irreducible and positive. Let z be any effective, irreducible curve which is a component of some representative of the class w. Then, it was shown in [32] that one can move the curve z to a boundary brane, we assume it is the first brane, and "absorb" it via a small instanton transition into the vector bundle V_1. This transition results in a new vector bundle V_1'. Henceforth, we will denote these two bundles by V and V' repectively. This process of absorbing all, or part, of the horizontal component of the bulk $M5$-brane curve, passing first through a torsion free sheaf (the small instanton) and then smoothing out to a vector bundle, was described in detail in [32]. As shown in [32], the effect of this transition, in addition to removing the z component of the bulk $M5$-brane curve, is to modify the spectral cover (5.3) describing V to a new spectral cover

$$\mathcal{C}' = n\sigma + \pi^* (\eta + z) \tag{5.4}$$

which describes vector bundle V'. This transition preserves the low energy gauge group H, but changes the third Chern class of the vector bundle and, hence, the number of families on the first 3-brane. Some of the physics of these chirality changing small instanton transitions are discussed in [32]. Finally, we note that it was demonstrated in [32] that this type of phase transition does not alter the line bundle $\mathcal{N}$ on the Calabi-Yau threefold.

Before continuing, we emphasize that it is also possible to have a small instanton transition in which a purely vertical component of the $M5$-brane curve is absorbed [32]. However, at least one of the bundles involved in such transitions is reducible and, hence, our previous discussion of moduli does not apply. We reserve discussion of such transitions for a later publication.

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5.1 Transition moduli

In the previous section, we showed how to compute the number of moduli, $n(V)$, of any positive, stable holomorphic SU(n) vector bundle over an arbitrary elliptically fibered Calabi-Yau threefold. For specificity, we computed $n(V)$ for $B = \mathbb{F}_r$. In this section, we are interested in the "new" moduli that appear when the vector bundle V on the first brane is struck by an $M5$-brane wrapped on a horizontal curve and makes a small instanton transition to bundle V'. We will refer to these as "transition" moduli. There are two issues to be discussed in this regard. The first is to compute the number, n_{tm}, of these transition moduli. The second is to elucidate the exact mathematical and physical relationship of the transition moduli to the torsion free sheaf and the horizontal curve. We begin by computing the number of transition moduli. This is clearly given by

$$n_{tm} = n(V') - n(V)\,. \tag{5.5}$$

Although this number can be computed on a Calabi-Yau threefold with any allowed base B, in this paper we will present the calculation for $B = \mathbb{F}_r$. The result follows immediately from equations (4.39) and (5.4). If we take

$$\eta = a\mathcal{S} + b\mathcal{E}\,, \qquad \eta + z = a'\mathcal{S} + b'\mathcal{E}\,, \tag{5.6}$$

where both associated spectral covers $\mathcal{C}$ and $\mathcal{C}'$ are assumed to be effective, irreducible and positive, then we can conclude the following.

- The number of transition moduli is given by

$$n_{tm} = n\left(a'b' - ab\right) - \left(n^2 - 2\right)\left(a' - a + b' - b\right) + \left(a' - a\right) r \left(\frac{n^2}{2} - 1\right) - \frac{nr}{2}\left(\left(a'\right)^2 - a^2\right). \tag{5.7}$$

Generically, this is a non-vanishing positive integer indicating that, at the time of the small instanton transition, new moduli are added to the holomorphic vector bundle. It is useful to illustrate these concepts by presenting several concrete examples.

Example 1. Consider the stable SU(3) vector bundle V specified in Example 1 at the end of section 3. In this case, $B = \mathbb{F}_1$, $G = \mathrm{SU}(3)$, $a = 7$, $b = 12$ and $\lambda = 1/2$. As shown in (3.52) and (4.40), $N_{\text{gen}} = 13$ and $n(V) = 104$ respectively. Using expressions (3.14), (3.35), (3.47), (4.18) and assuming that the vector bundle V_2 over the second end-of-the-world brane is trivial, we find from (5.1) and (5.2) that the bulk space $M5$-brane is non-vanishing and wrapped on a holomorphic curve $W \subset X$ where

$$W_B = \pi^*(17\mathcal{S} + 20\mathcal{E})\,, \qquad a_f = 100\,. \tag{5.8}$$

Note that W is an effective class in $H_2(X,\mathbb{Z})$, as it must be. Using the results of [14], one can show that there is always a region of the moduli space of the $M5$-brane where

$$z = \mathcal{E} \subset 17\mathcal{S} + 20\mathcal{E} \tag{5.9}$$

is an effective subcurve. Curve $z = \mathcal{E}$ can move to the first end-of-the-world brane, inducing a small instanton transition to a new vector bundle V' specified by $B = \mathbb{F}_1$, $G = SU(3)$, $a' = 7$, $b' = 13$ and $\lambda = 1/2$, where we have used (5.4). But, this is precisely the stable $SU(3)$ vector bundle V' presented in Example 2 at the end of section 3. As shown in (3.56) and (4.41), for this bundle $N'_{\text{gen}} = 17$ and $n(V') = 118$. Using (5.7), we find that the number of transition moduli is

$$n_{tm} = 14 . \tag{5.10}$$

That is, the small instanton transition with $M5$-brane curve $z = \mathcal{E}$ increases the number of vector bundle moduli from $n(V) = 104$ before the collision to $n(V') = 118$ after it. Note, in passing, that this transition also changes the number of generations from $N_{\text{gen}} = 13$ to $N'_{\text{gen}} = 17$.

Example 2. Consider again the stable $SU(3)$ vector bundle V specified in Example 1 at the end of section 3, with $B = \mathbb{F}_1$, $G = SU(3)$, $a = 7$, $b = 12$, $\lambda = 1/2$ and $N_{\text{gen}} = 13$, $n(V) = 104$. Assuming that the vector bundle V_2 over the second end-of-the-world brane is trivial, the bulk space $M5$-brane is non-vanishing and wrapped on the holomorphic curve W specified in (5.8). Using the results of [14], one can show that there is always another region of the moduli space of the $M5$-brane where

$$z = \mathcal{S} \subset 17\mathcal{S} + 20\mathcal{E} \tag{5.11}$$

is an effective subcurve. Curve $z = \mathcal{S}$ can move to the first end-of-the-world brane, inducing a small instanton transition to a new vector bundle V' specified by $B = \mathbb{F}_1$, $G = SU(3)$, $a' = 8$, $b' = 12$ and $\lambda = 1/2$, where we have used (5.4). But, this is precisely the stable $SU(3)$ vector bundle V' presented in Example 3 at the end of section 3. As shown in (3.60) and (4.42), for this bundle $N'_{\text{gen}} = 16$ and $n(V') = 114$. Using (5.7), we find that the number of transition moduli is

$$n_{tm} = 10 . \tag{5.12}$$

That is, the small instanton transition with $M5$-brane curve $z = \mathcal{S}$ increases the number of vector bundle moduli from $n(V) = 104$ before the collision to $n(V') = 114$ after it. Note, in passing, that this transition also changes the number of generations from $N_{\text{gen}} = 13$ to $N'_{\text{gen}} = 16$.

We now turn to the second important question regarding transion moduli. That is, what is the exact relationship of the transition moduli to the torsion free sheaf and the horizontal curve?

5.2 Localization of the transition moduli on $\pi^* z$

One could well speculate that the transition moduli arise as the moduli of the spectral cover $\mathcal{C}'$ restricted to the lift of the horizontal curve, $\pi^* z$. We claim that this is, in fact, correct, as we now demonstrate for the case of $B = \mathbb{F}_r$. For a divisor D in X, there is a short exact sequence

$$0 \to E \otimes \mathcal{O}_X(-D) \to E \to E|_D \to 0 , \tag{5.13}$$

where E is any holomorphic vector bundle on X. Taking E to be trivial, $E = \mathcal{O}$, implies that

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0\,. \tag{5.14}$$

We begin by choosing

$$z = \mathcal{E} \tag{5.15}$$

and

$$D = \pi^* \mathcal{E}\,, \tag{5.16}$$

which is a divisor in X. Then exact sequence (5.14) becomes

$$0 \to \mathcal{O}_X(-\pi^*\mathcal{E}) \to \mathcal{O}_X \to \mathcal{O}_{\pi^*\mathcal{E}} \to 0\,. \tag{5.17}$$

Tensoring each term with the line bundle $\mathcal{O}_X(n\sigma + \pi^*(a\mathcal{S} + (b+1)\mathcal{E}))$ yields

$$0 \to L \to L' \to L'|_{\pi^*\mathcal{E}} \to 0\,, \tag{5.18}$$

where

$$L = \mathcal{O}_X(\mathcal{C})\,, \qquad L' = \mathcal{O}_X(\mathcal{C}') \tag{5.19}$$

and

$$\mathcal{C} = n\sigma + \pi^*(a\mathcal{S} + b\mathcal{E})\,, \qquad \mathcal{C}' = n\sigma + \pi^*(a\mathcal{S} + (b+1)\mathcal{E})\,. \tag{5.20}$$

Note that $L'|_{\pi^*\mathcal{E}}$ is the line bundle associated with the restriction of $\mathcal{C}'$ to $\pi^* z$ and, hence, is the object of interest in the conjecture.

In passing, note that

$$L'|_{\pi^*\mathcal{E}} = L|_{\pi^*\mathcal{E}}\,. \tag{5.21}$$

To see this, use the fact

$$L' = L \otimes \mathcal{O}_X(\pi^*\mathcal{E}) \tag{5.22}$$

and, hence, that

$$L'|_{\pi^*\mathcal{E}} = L|_{\pi^*\mathcal{E}} \otimes \mathcal{O}_X(\pi^*\mathcal{E})|_{\pi^*\mathcal{E}}\,. \tag{5.23}$$

But

$$c_1(\mathcal{O}_X(\pi^*\mathcal{E})|_{\pi^*\mathcal{E}}) = \pi^*\mathcal{E}|_{\pi^*\mathcal{E}} = \pi^*\mathcal{E} \cdot \pi^*\mathcal{E}\,, \tag{5.24}$$

which vanishes using equations (3.14) and (3.33). It follows that

$$\mathcal{O}_X(\pi^*\mathcal{E})|_{\pi^*\mathcal{E}} = \mathcal{O}_{\pi^*\mathcal{E}}\,, \tag{5.25}$$

from which expression (5.21) follows. This is a result of the fact that $\mathcal{E} \cdot \mathcal{E} = 0$ and will not be true for the lift of curve $\mathcal{S}$.

It is useful to note that

$$\pi^*\mathcal{E} = K3\,. \tag{5.26}$$

To prove this, recall that K3 is defined as a twofold elliptically fibered over $\mathbb{P}^1$ with 24 singular fibers. Now consider $\pi^*\mathcal{E}$, which is indeed a twofold elliptically fibered over $\mathcal{E} = \mathbb{P}^1$.

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How many singular fibers does it have? Note that the line bundle of the discriminant curve of base B is

$$\Delta = -12K_B\,, \tag{5.27}$$

where K_B is the canonical bundle on B. It follows that

$$c_1(\Delta) = 12c_1(B)\,, \tag{5.28}$$

where we have used equation (3.30). For $B = \mathbb{F}_r$, we see from (3.35) that the discriminant curve in $\mathbb{F}_r$ is

$$\mathcal{D} = 24\mathcal{S} + 12(r+2)\mathcal{E}\,. \tag{5.29}$$

Therefore, the number of discriminant points in $\mathcal{E} = \mathbb{P}^1$ is

$$\mathcal{D}\cdot\mathcal{E} = 24\mathcal{S}\cdot\mathcal{E} + 12(r+2)\mathcal{E}\cdot\mathcal{E} = 24\,, \tag{5.30}$$

where we have used (3.14). It follows that $\pi^*\mathcal{E}$ is a K3 surface. Henceforth, we will denote $\pi^*\mathcal{E}$ by K3.

The short exact sequence (5.18) is now written as

$$0 \to L \to L' \to L'|_{K3} \to 0\,. \tag{5.31}$$

This implies, in the usual way, a long exact sequence given in part by

$$0 \to H^0(X,L) \to H^0(X,L') \to H^0(K3, L'|_{K3}) \to H^1(X,L) \to\ . \tag{5.32}$$

Since $\mathcal{C}$ is positive, it follows from the Kodaira vanishing theorem (4.12) that

$$H^1(X,L) = 0\,. \tag{5.33}$$

Hence

$$0 \to H^0(X,L) \to H^0(X,L') \to H^0(K3, L'|_{K3}) \to 0\,, \tag{5.34}$$

which implies that

$$H^0(K3, L'|_{K3}) = \frac{H^0(X,L')}{H^0(X,L)}\,. \tag{5.35}$$

Therefore

$$h^0(K3, L'|_{K3}) = h^0(X,L') - h^0(X,L) \tag{5.36}$$

and, hence

$$h^0(K3, L'|_{K3}) = n(V') - n(V) = n_{tm}\,. \tag{5.37}$$

We conclude that the transition moduli are precisely the moduli associated with $\mathcal{O}_X(\mathcal{C}')|_{K3}$ or, equivalently, the restriction of spectral cover $\mathcal{C}'$ to the lift of $z = \mathcal{E}$, as claimed.

This result can be obtained by direct computation, as we now show. In this context, the Riemann-Roch theorem states that

$$h^0(K3, L|_{K3}) - h^1 + h^2 = \chi_E(K3, L|_{K3})\,, \tag{5.38}$$

where $\chi_E(K3, L|_{K3})$ is the Euler characteristic and h^q for $q = 1, 2$ are the dimensions of the higher cohomology groups of $K3$ evaluated in $L|_{K3}$. To proceed, we must find the conditions under which the line bundle $L|_{K3}$ is positive. Recall from (5.19) and (5.20) that

$$L = \mathcal{O}_X(n\sigma + \pi^*(a\mathcal{S} + b\mathcal{E})) \tag{5.39}$$

and, hence, that

$$L|_{K3} = \mathcal{O}_{K3}(n\sigma|_{K3} + a(\pi^*\mathcal{S})|_{K3} + b(\pi^*\mathcal{E})|_{K3}) \,. \tag{5.40}$$

Using the facts that

$$\pi^*\mathcal{S}|_{K3} = \pi^*\mathcal{S} \cdot \pi^*\mathcal{E} = \pi^*(\mathcal{S} \cdot \mathcal{E}) = \pi^*(1) = F \tag{5.41}$$

and

$$\pi^*\mathcal{E}|_{K3} = \pi^*\mathcal{E} \cdot \pi^*\mathcal{E} = \pi^*(\mathcal{E} \cdot \mathcal{E}) = 0 \,, \tag{5.42}$$

where we have used (3.14) and (3.33), it follows that

$$L|_{K3} = \mathcal{O}_{K3}(\mathcal{C}|_{K3}) \,, \tag{5.43}$$

where

$$\mathcal{C}|_{K3} = n\sigma|_{K3} + aF \,. \tag{5.44}$$

We know from the discussion in section 3 that $L|_{K3}$ is positive if and only if $\mathcal{C}|_{K3} \cdot c > 0$ for every effective curve c in $K3$. It is easy to see that the basis of effective curves in $K3$ is given by

$$\sigma|_{K3} \,, \qquad F \,. \tag{5.45}$$

First consider $\mathcal{C}|_{K3} \cdot F$. It follows from the fact

$$\sigma \cdot F = 1 \,, \qquad F \cdot F = 0 \tag{5.46}$$

that

$$\mathcal{C}|_{K3} \cdot F = n \,, \tag{5.47}$$

which is always positive. Now consider $\mathcal{C}|_{K3} \cdot \sigma|_{K3}$. To evaluate this, note, using (3.23), that

$$\sigma|_{K3} \cdot \sigma|_{K3} = -\pi^*(c_1(\mathbb{F}_r)) \cdot \sigma \cdot \pi^*\mathcal{E} \,. \tag{5.48}$$

Then, using (3.14), (3.33), (3.35) and (5.46) we find that

$$\mathcal{C}|_{K3} \cdot \sigma|_{K3} = -2n + a \,, \tag{5.49}$$

which is positive if and only if

$$a > 2n \,. \tag{5.50}$$

However, the positivity of $\mathcal{C}$ already demands that $a > 2n$, as stated in (4.38). Therefore, we conclude that $L|_{K3}$ and, hence, $\mathcal{C}|_{K3}$ is positive. This allows us to employ the Kodaira vanishing theorem to evaluate the higher cohomology groups. A straightforward extension of the discussion in section 4 allows us to conclude that

$$H^q(K3, L|_{K3}) = 0 \tag{5.51}$$

for $q > 0$. It follows that $h^q = 0$ for $q = 1, 2$ and the Riemann-Roch theorem (5.38) simplifies to

$$h^0(K3, L|_{K3}) = \chi_E(K3, L|_{K3}) \,. \tag{5.52}$$

The Euler characteristic is determined from the Atiyah-Singer index theorem to be

$$\chi_E(K3, L|_{K3}) = \int_{K3} ch(\mathcal{O}_{K3}(\mathcal{C}|_{K3})) \wedge Td(TK3) \,, \tag{5.53}$$

which, on the Calabi-Yau twofold $K3$ takes the form

$$\chi_E(K3, L|_{K3}) = \frac{1}{2} \int_{K3} c_1^2(\mathcal{O}_{K3}(\mathcal{C}|_{K3})) + \frac{1}{12} \int_{K3} c_2(TK3) \,. \tag{5.54}$$

Then, using (5.44) and the fact that

$$c_2(TK3) = 24 \,, \tag{5.55}$$

we find

$$h^0(K3, L|_{K3}) = 2 + an - n^2 \,. \tag{5.56}$$

We conclude by noting that for $a' = a$ and $b' = b + 1$, equation (5.7) becomes

$$n_{tm} = 2 + an - n^2 \,. \tag{5.57}$$

It follows from this and the relation (5.21) that

$$h^0(K3, L'|_{K3}) = n_{tm} \,, \tag{5.58}$$

as in (5.37) .

Having proven the claim for $z = \mathcal{E}$, we now extend our result to small instanton transitions with curve $z = \mathcal{S}$. The exact sequence of bundles (5.31) now reads

$$0 \to L \to L' \to L'|_{\pi^* \mathcal{S}} \to 0 \,, \tag{5.59}$$

where

$$L = \mathcal{O}_X(\mathcal{C}) \,, \qquad L' = \mathcal{O}_X(\mathcal{C}') \tag{5.60}$$

and

$$\mathcal{C} = n\sigma + \pi^*(a\mathcal{S} + b\mathcal{E}) \,, \qquad \mathcal{C}' = n\sigma + \pi^*((a+1)\mathcal{S} + b\mathcal{E}) \,. \tag{5.61}$$

Information about the surface $\pi^*\mathcal{S}$, elliptically fibered over $\mathcal{S} = \mathbb{P}^1$, can be obtained by counting the number of degenerate fibers. As explained above, this number can be found by intersecting the discriminant curve $\mathcal{D}$ given in equation (5.29) with the curve $\mathcal{S}$ on the base. Using equation (3.14), one finds

$$\mathcal{D} \cdot \mathcal{S} = 12(2 - r) \,. \tag{5.62}$$

When $r = 0$, we see that the surface $\pi^*\mathcal{S}$ has 24 degenerate fibers implying that $\pi^*\mathcal{S} = K3$. For $r = 1$, the number of degenerate fibers is 12 and, hence, $\pi^*\mathcal{S}$ is the del Pezzo surface dP_9, while for $r = 2$ the resulting surface is a nonsingular fibration. One can show that this fibration is simply the cross product $\mathbb{P}^1 \times T^2$.

As before, the exact sequence of bundles (5.59) produces the exact sequence of cohomology groups

$$0 \to H^0(X, L) \to H^0(X, L') \to H^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) \to 0\,, \tag{5.63}$$

where we have used the vanishing of $H^1(X, L)$ discussed above. Therefore

$$h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) = h^0(X, L') - h^0(X, L) \tag{5.64}$$

and, hence

$$h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) = n(V') - n(V) = n_{tm}\,. \tag{5.65}$$

In analogy with the previous case, we see that the transition moduli are associated with the moduli of the bundle $\mathcal{O}_X(\mathcal{C}')|_{\pi^*\mathcal{S}}$ or, equivalently, the restriction of the spectral cover $\mathcal{C}'$ to the lift of the $z = \mathcal{S}$ curve.

For completeness, we now compute $h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}})$ by an independent calculation and demonstrate that the result is identical to n_{tm} found from the general formula (5.7). Using equation (3.14), we get

$$\pi^*\mathcal{S}|_{\pi^*\mathcal{S}} = \pi^*(\mathcal{S}\cdot\mathcal{S}) = -rF\,, \qquad \pi^*\mathcal{E}|_{\pi^*\mathcal{S}} = \pi^*(\mathcal{E}\cdot\mathcal{S}) = F \tag{5.66}$$

and, therefore, the bundle L' restricted to $\pi^*\mathcal{S}$ is given by

$$L'|_{\pi^*\mathcal{S}} = \mathcal{O}_{\pi^*\mathcal{S}}(\mathcal{C}'|_{\pi^*\mathcal{S}})\,, \tag{5.67}$$

where

$$\mathcal{C}'|_{\pi^*\mathcal{S}} = n\sigma|_{\pi^*\mathcal{S}} + (b - (a+1)r)F\,. \tag{5.68}$$

Here $\sigma|_{\pi^*\mathcal{S}}$ represents the global section of the $\pi^*\mathcal{S}$ elliptic fibration. As before, we would like to use the Riemann-Roch theorem to compute $h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}})$. From the previous analysis, it follows that for this to be straightforward the higher cohomology groups have to vanish

$$H^q(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) = 0\,, \qquad q > 0\,. \tag{5.69}$$

This last equation is guaranteed by the Kodaira vanishing theorem (4.8) provided the bundle L' restricted to $\pi^*\mathcal{S}$ can be represented as

$$L'|_{\pi^*\mathcal{S}} = K_{\pi^*\mathcal{S}} \otimes L''_{\pi^*\mathcal{S}}\,, \tag{5.70}$$

where $K_{\pi^*\mathcal{S}}$ is the canonical bundle of $\pi^*\mathcal{S}$ and $L''_{\pi^*\mathcal{S}}$ is some positive line bundle. Note that, unlike the $z = \mathcal{E}$ case, surface $\pi^*\mathcal{S}$ is not a Calabi-Yau twofold (unless r=0) and, hence, the canonical bundle $K_{\pi^*\mathcal{S}}$ is nontrivial. In fact, using the adjunction formula we see that

$$K_{\pi^*\mathcal{S}} = K_X \otimes \mathcal{O}_X(\pi^*\mathcal{S})|_{\pi^*\mathcal{S}} = \mathcal{O}_X(\pi^*\mathcal{S})|_{\pi^*\mathcal{S}}\,, \tag{5.71}$$

where we have used that fact that K_X is trivial. But

$$c_1(\mathcal{O}_X(\pi^*\mathcal{S})|_{\pi^*\mathcal{S}}) = \pi^*(\mathcal{S}\cdot\mathcal{S}) = -rF\,. \tag{5.72}$$

It follows that

$$K_{\pi^*\mathcal{S}} = \mathcal{O}_{\pi^*\mathcal{S}}(-rF)\,. \tag{5.73}$$

Using equations (5.67), (5.68), (5.70) and (5.73), we conclude that the line bundle $L''_{\pi^*\mathcal{S}}$ is of the form

$$L''_{\pi^*\mathcal{S}} = \mathcal{O}_{\pi^*\mathcal{S}}(\mathcal{C}''_{\pi^*\mathcal{S}})\,, \tag{5.74}$$

where

$$\mathcal{C}''_{\pi^*\mathcal{S}} = n\sigma|_{\pi^*\mathcal{S}} + (b-ar)F\,. \tag{5.75}$$

Let us find under what conditions $L''_{\pi^*\mathcal{S}}$ is positive. As before, we have to require that $\mathcal{C}''_{\pi^*\mathcal{S}}\cdot c > 0$ for all effective curves c which, in this case, have the basis

$$\sigma|_{\pi^*\mathcal{S}}\,, \qquad F\,. \tag{5.76}$$

By using relations (3.23), (5.46) and the fact that

$$\sigma|_{\pi^*\mathcal{S}}\cdot\sigma|_{\pi^*\mathcal{S}} = -\pi^*(c_1(\mathbb{F}_r))\cdot\sigma\cdot\pi^*\mathcal{S} = -(2-r)\,, \tag{5.77}$$

we find that the line bundle $L''_{\pi^*\mathcal{S}}$ is positive if and only if

$$b > ar + (2-r)n\,. \tag{5.78}$$

But condition (5.78) is just the positivity condition (3.37). Therefore, if the original spectral cover $\mathcal{C}$ is chosen to be positive the line bundle $L''_{\pi^*\mathcal{S}}$ is automatically positive. As a consequence, equation (5.69) is valid and $h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}})$ coincides with the Euler characteristic

$$h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) = \chi_E(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}})\,. \tag{5.79}$$

The latter can be computed by using the index theorem

$$\chi_E(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) = \int_{\pi^*\mathcal{S}} ch(\mathcal{O}_{\pi^*\mathcal{S}}(\mathcal{C}'|_{\pi^*\mathcal{S}}))\wedge Td(T\pi^*\mathcal{S}) \tag{5.80}$$

which, on the twofold $\pi^*\mathcal{S}$, takes the form

$$\begin{aligned}\chi_E(\pi^*\mathcal{S}, L'_{\pi^*\mathcal{S}}) \;=\; & \frac{1}{2}\int_{\pi^*\mathcal{S}} c_1^2(\mathcal{O}_{\pi^*\mathcal{S}}(\mathcal{C}'|_{\pi^*\mathcal{S}})) + \frac{1}{2}\int_{\pi^*\mathcal{S}} c_1(\mathcal{O}_{\pi^*\mathcal{S}}(\mathcal{C}'|_{\pi^*\mathcal{S}}))\wedge c_1(T\pi^*\mathcal{S}) + \\ & + \frac{1}{12}\int_{\pi^*\mathcal{S}}(c_2(T\pi^*\mathcal{S}) + c_1^2(T\pi^*\mathcal{S}))\,. \end{aligned} \tag{5.81}$$

The first Chern class, $c_1(T\pi^*\mathcal{S})$, is given by

$$c_1(T\pi^*\mathcal{S}) = -c_1(K_{\pi^*\mathcal{S}}) = rF\,, \tag{5.82}$$

where we have used (5.71). Note that

$$c_1^2(T\pi^*\mathcal{S}) = 0\,. \tag{5.83}$$

Finding $c_2(T\pi^*\mathcal{S})$ requires some work. Consider the short exact sequence

$$0 \to T\pi^*\mathcal{S} \to TX|_{\pi^*\mathcal{S}} \to N\pi^*\mathcal{S} \to 0\,, \tag{5.84}$$

where $N\pi^*\mathcal{S}$ is the normal bundle. The sequence (5.84) follows directly from the definition of the normal bundle

$$N\pi^*\mathcal{S} = \frac{TX|_{\pi^*\mathcal{S}}}{T\pi^*\mathcal{S}}\,. \tag{5.85}$$

On the other hand, we can find the normal bundle by relating it to the line bundle $\mathcal{O}_X(\pi^*\mathcal{S})|_{\pi^*\mathcal{S}}$ using

$$N\pi^*\mathcal{S} = \mathcal{O}_X(\pi^*\mathcal{S})|_{\pi^*\mathcal{S}} = K_{\pi^*\mathcal{S}} = \mathcal{O}_{\pi^*\mathcal{S}}(-rF)\,, \tag{5.86}$$

where equations (5.71) and (5.73) have been used. Exact sequence (5.84) then produces the following relation for total Chern classes

$$c(TX|_{\pi^*\mathcal{S}}) = c(T\pi^*\mathcal{S}) \wedge c(\mathcal{O}_{\pi^*\mathcal{S}}(-rF))\,. \tag{5.87}$$

In particular, we have

$$c_2(T\pi^*\mathcal{S}) = c_2(TX|_{\pi^*\mathcal{S}})\,. \tag{5.88}$$

Using expression (4.18) for $c_2(TX)$, we conclude that

$$c_2(T\pi^*\mathcal{S}) = 12\sigma \cdot \pi^*(c_1(\mathbb{F}_r)) \cdot \pi^*\mathcal{S} = 12(2-r)\,. \tag{5.89}$$

One can now calculate the Euler characteristic (5.81) and, therefore, $h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}})$. The result is

$$h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) = nb - (n^2-2) + \frac{rn}{2}(n-1) - r(na+1)\,. \tag{5.90}$$

On the other hand, equation (5.7) with $a' = a+1$, $b' = b$ becomes

$$n_{tm} = nb - (n^2-2) + \frac{rn}{2}(n-1) - r(na+1)\,. \tag{5.91}$$

We see that equations (5.90) and (5.91) coincide and, hence that

$$h^0(\pi^*\mathcal{S}, L'|_{\pi^*\mathcal{S}}) = n_{tm} \tag{5.92}$$

as in (5.65).
Using (5.37) and (5.65), it is now straightforward to prove the following.

- The transition moduli arise as the moduli of the spectral cover $\mathcal{C}'$ restricted to the lift, π^*z, of the horizontal curve z, for any effective curve z. Specifically,

$$h^0(\pi^*z, L'|_{\pi^*z}) = n_{tm}\,, \tag{5.93}$$

  where n_{tm} is given in expression (5.7).

To be concrete, let us give several examples.

Example 1. Consider the small instanton transition presented in Example 1 earlier in this section. Recall that the pre-transition vector bundle V was specified by $B = \mathbb{F}_1$, $G = SU(3)$, $a = 7$, $b = 12$ and $\lambda = 1/2$. In this case, the $M5$-brane involved in the transition is wrapped on the curve

$$z = \mathcal{E}\,, \tag{5.94}$$

$\pi^*\mathcal{E} = K3$ and

$$L'|_{K3} = \mathcal{O}_{K3}(3\sigma|_{K3} + 7F)\,, \tag{5.95}$$

where we have used (5.44). Then, we conclude from (5.10) and (5.93) that

$$h^0(K3, L'|_{K3}) = 14 \tag{5.96}$$

is the number of transition moduli.

Example 2. Consider the small instanton transition presented in Example 2 earlier in this section. Recall that the pre-transition vector bundle V was again specified by $B = \mathbb{F}_1$, $G = SU(3)$, $a = 7$, $b = 12$ and $\lambda = 1/2$. In this case, however, the $M5$-brane involved in the transition is wrapped on the curve

$$z = \mathcal{S}\,, \tag{5.97}$$

$\pi^*\mathcal{S} = dP_9$ and

$$L'|_{dP_9} = \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F)\,, \tag{5.98}$$

where we have used (5.68). We conclude from (5.12) and (5.93) that

$$h^0(dP_9, L'|_{dP_9}) = 10 \tag{5.99}$$

is the number of transition moduli.

6. Evaluation of transition moduli on the curve z

In the previous section, we have presented an interpretation of the transition moduli as the moduli of the spectral cover $\mathcal{C}'$ restricted to π^*z. However, it is clearly of interest to ask whether the transition moduli can be related directly to intrinsic quantities on the curve $\pi^*z \cdot \sigma$ alone. The answer to this is in the affirmative, as we will now demonstrate.

As above, we begin by restricting the discussion to the horizontal curve $z = \mathcal{E}$. Recall from (5.21) and (5.43), (5.44) that the line bundle associated with the transition moduli is $L'|_{K3} = L|_{K3}$, which we now denote by $\mathcal{L}_a$ to streamline our notation. That is

$$\mathcal{L}_a = \mathcal{O}_{K3}(n\sigma|_{K3} + aF). \tag{6.1}$$

Note that this can be written as the tensor product

$$\mathcal{L}_a = \mathcal{L}_0 \otimes \pi^*\mathcal{O}_{\mathcal{E}}(a)\,, \tag{6.2}$$

where

$$\mathcal{L}_0 = \mathcal{O}_{K3}(n\sigma|_{K3})\,. \tag{6.3}$$

We now want to consider the push-forward of the line bundle $\mathcal{L}_a$ onto the curve $z = \mathcal{E}$ given by

$$\pi_* \mathcal{L}_a = \pi_* (\mathcal{L}_0 \otimes \pi^* \mathcal{O}_{\mathcal{E}}(a)) \,. \tag{6.4}$$

Using the projection formula

$$\pi_* (\mathcal{F} \otimes \pi^* \mathcal{G}) = (\pi_* \mathcal{F}) \otimes \mathcal{G} \tag{6.5}$$

for arbitrary bundles $\mathcal{F}$ and $\mathcal{G}$, equation (6.4) becomes

$$\pi_* \mathcal{L}_a = \pi_* \mathcal{L}_0 \otimes \mathcal{O}_{\mathcal{E}}(a) \,. \tag{6.6}$$

Now, since the spectral cover $\mathcal{C}'|_{K3}$ is an n-fold cover of the base curve $\mathcal{E}$, it follows that the direct image of any line bundle on $\mathcal{C}'|_{K3}$ onto $\mathcal{E}$ is a rank n vector bundle. At this point, for specificity, it is convenient to choose a value of n, which we take to be

$$n = 3 \,. \tag{6.7}$$

Our remarks will remain true for any value of n. It follows that we can always represent $\pi_* \mathcal{L}_0$ as

$$\pi_* \mathcal{L}_0 = \mathcal{O}_{\mathcal{E}}(A) \oplus \mathcal{O}_{\mathcal{E}}(B) \oplus \mathcal{O}_{\mathcal{E}}(C) \tag{6.8}$$

for some integers A, B and C. Then from (6.6) we have

$$\pi_* \mathcal{L}_a = (\mathcal{O}_{\mathcal{E}}(A) \oplus \mathcal{O}_{\mathcal{E}}(B) \oplus \mathcal{O}_{\mathcal{E}}(C)) \otimes \mathcal{O}_{\mathcal{E}}(a) \tag{6.9}$$

and, hence, that

$$\pi_* \mathcal{L}_a = \mathcal{O}_{\mathcal{E}}(a + A) \oplus \mathcal{O}_{\mathcal{E}}(a + B) \oplus \mathcal{O}_{\mathcal{E}}(a + C) \,. \tag{6.10}$$

We now need to compute the values of integers A, B and C.

One can show, using a Leray spectral sequence, that for any value of n and a

$$H^0(\mathcal{E}, \pi_* \mathcal{L}_a) = H^0(K3, \mathcal{L}_a) \,. \tag{6.11}$$

Here, we will continue to assume that $n = 3$. Let us first consider the case where $a = 0$ and, therefore, $\mathcal{L}_a = \mathcal{L}_0 = \mathcal{O}_{K3}(3\sigma|_{K3})$. Now, it follows from (5.49) that $\sigma|_{K3}^2 = -2$ and, hence, that the curve $3\sigma|_{K3}$ is isolated. Therefore,

$$h^0(K3, \mathcal{L}_0) = 1 \,, \tag{6.12}$$

which implies, using (6.11), that

$$h^0(\mathcal{E}, \pi_* \mathcal{L}_0) = 1 \,. \tag{6.13}$$

However, line bundles $\mathcal{O}_{\mathbb{P}^1}(m)$ over a projective line $\mathbb{P}^1$ have the generic property that

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = m + 1 \tag{6.14}$$

for $m \geq 0$ and

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = 0 \tag{6.15}$$

for $m < 0$. Therefore, putting together (6.8), (6.13), (6.14) and (6.15) we see, without loss of generality, that we can choose

$$A = 0\,, \qquad B, C < 0\,. \tag{6.16}$$

Second, let us consider the case where $a > 6$. Since, for $n = 3$, this satisfies the positivity condition (5.50), then it follows from (5.56) and (6.11) that

$$h^0(\mathcal{E}, \pi_* \mathcal{L}_a) = 3a - 7\,. \tag{6.17}$$

The reader can easily verify that, since A, B and C are fixed coefficients, one can only solve (6.17) for arbitrary values of $a > 6$ if

$$a + B \geq 0\,, \qquad a + C \geq 0\,. \tag{6.18}$$

It then follows from (6.10) and (6.14) that

$$h^0(\mathcal{E}, \pi_* \mathcal{L}_a) = (a + 1) + (a + B + 1) + (a + C + 1)\,. \tag{6.19}$$

But, from (6.17) this must equal $3a - 7$, from which we find that

$$B + C = -10\,. \tag{6.20}$$

Third, let us now find the formula for $h^0(K3, \mathcal{L}_a)$, and, hence, using (6.11) for $h^0(\mathcal{E}, \pi_* \mathcal{L}_a)$, for the remaining values of a, $0 < a \leq 6$. Note that $\sigma|_{K3}$ is a divisor in $K3$. Let $\mathcal{D} \subset K3$ be any divisor and $E = \mathcal{O}_{K3}(\mathcal{D})$ the associated line bundle. Then, it follows from (5.13) that there is the short exact sequence

$$0 \to \mathcal{O}_{K3}(\mathcal{D} - \sigma|_{K3}) \to \mathcal{O}_{K3}(\mathcal{D}) \to \mathcal{O}_{\sigma|_{K3}}(\mathcal{D} \cdot \sigma|_{K3}) \to 0\,, \tag{6.21}$$

where we have used $\mathcal{D}|_{K3} = \mathcal{D} \cdot \sigma|_{K3}$. This then implies the long exact sequence of cohomology groups given by

$$\begin{aligned} 0 &\to H^0(K3, \mathcal{O}_{K3}(\mathcal{D} - \sigma|_{K3})) \to H^0(K3, \mathcal{O}_{K3}(\mathcal{D})) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D} \cdot \sigma|_{K3})) \to \\ &\to H^1(K3, \mathcal{O}_{K3}(\mathcal{D} - \sigma|_{K3})) \to\ . \end{aligned} \tag{6.22}$$

In the appropriate places, we have inserted the fact that $\sigma|_{K3} = \mathbb{P}^1$. First, assume that $\mathcal{D} \cdot \sigma|_{K3} < 0$. Then, it follows from (6.15) that $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D} \cdot \sigma|_{K3})) = 0$ and, hence

$$h^0(K3, \mathcal{O}_{K3}(\mathcal{D})) = h^0(K3, \mathcal{O}_{K3}(\mathcal{D} - \sigma|_{K3}))\,, \qquad \mathcal{D} \cdot \sigma|_{K3} < 0\,. \tag{6.23}$$

Now assume $\mathcal{D} \cdot \sigma|_{K3} \geq 0$. Equation (6.14) then implies

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D} \cdot \sigma|_{K3})) = \mathcal{D} \cdot \sigma|_{K3} + 1\,. \tag{6.24}$$

Note, using $\sigma|_{K3}^2 = -2$, that $(\mathcal{D} - \sigma|_{K3}) \cdot \sigma|_{K3} > 0$. If we further assume that $\mathcal{D} \cdot F > 1$, then $(\mathcal{D} - \sigma|_{K3}) \cdot F > 0$ and, hence, $\mathcal{D} - \sigma|_{K3}$ is a positive divisor in $K3$. It then follows from the Kodaira vanishing theorem that

$$H^q(K3, \mathcal{O}_{K3}(\mathcal{D} - \sigma|_{K3})) = 0 \tag{6.25}$$

for $q > 0$. Hence, the exact sequence (6.22) truncates to

$$0 \to H^0(K3, \mathcal{O}_{K3}(\mathcal{D} - \sigma|_{K3})) \to H^0(K3, \mathcal{O}_{K3}(\mathcal{D})) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D} \cdot \sigma|_{K3})) \to 0\,. \quad (6.26)$$

Therefore, using (6.24), for $\mathcal{D} - \sigma|_{K3} > 0$ we have

$$h^0(K3, \mathcal{O}_{K3}(\mathcal{D})) - h^0(K3, \mathcal{O}_{K3}(\mathcal{D} - \sigma|_{K3})) = \mathcal{D} \cdot \sigma|_{K3} + 1\,, \qquad \mathcal{D} \cdot \sigma|_{K3} \geq 0\,. \quad (6.27)$$

Now, from equations (6.23) and (6.27) we can compute $h^0(K3, \mathcal{O}_{K3}(k\sigma|_{K3} + aF))$ for $k = 0, 1, 2, 3$. Start with $k = 0$. In this case, equations (6.11) and (6.14) imply that

$$h^0(K3, \mathcal{O}_{K3}(aF)) = h^0(K3, \pi^*\mathcal{O}_{\mathcal{E}}(a)) = h^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(a)) = a + 1\,, \qquad a \geq 0\,. \quad (6.28)$$

Now take $k = 1$. Any effective divisor of the class $\sigma|_{K3} + aF$ has intersection number 1 with F and, hence, consists of some section σ' of $K3$ plus fibers. A special property of elliptically fibered $K3$ twofolds is that every section σ' satisfies $(\sigma')^2 = -2$ and, therefore, is rigid. Thus, every point in the projective space $\mathbb{P}H^0(K3, \mathcal{O}_{K3}(\sigma|_{K3} + aF))$ corresponds to some σ' plus fibers, and none of the σ' can move continuously. Therefore, all the σ' must be equal to one another and, hence, to the rigid section $\sigma|_{K3}$. We conclude that the only freedom to move the divisor is in the fibers, so we get

$$h^0(K3, \mathcal{O}_{K3}(\sigma|_{K3} + aF)) = h^0(K3, \mathcal{O}_{K3}(aF)) = a + 1\,, \qquad a \geq 0\,. \quad (6.29)$$

As the next step, take $k = 2$ and set $\mathcal{D} = 2\sigma|_{K3} + aF$. First, note that

$$(2\sigma|_{K3} + aF) \cdot \sigma|_{K3} = a - 4\,. \quad (6.30)$$

This breaks into two cases. For $0 \leq a \leq 3$, $\mathcal{D} \cdot \sigma|_{K3} < 0$. Then, it follows from (6.23) and (6.29) that

$$h^0(K3, \mathcal{O}_{K3}(2\sigma|_{K3} + aF)) = h^0(K3, \mathcal{O}_{K3}(\sigma|_{K3} + aF)) = a + 1\,, \qquad 0 \leq a \leq 3\,. \quad (6.31)$$

Now let $a \geq 4$. In this case, $\mathcal{D} \cdot \sigma|_{K3} \geq 0$ and

$$(\sigma|_{K3} + aF) \cdot \sigma|_{K3} = a - 2 > 0\,, \qquad (\sigma|_{K3} + aF) \cdot F = 1 > 0\,. \quad (6.32)$$

Then equation (6.27) is applicable and implies that

$$h^0(K3, \mathcal{O}_{K3}(2\sigma|_{K3} + aF)) = h^0(K3, \mathcal{O}_{K3}(\sigma|_{K3} + aF)) + a - 3 = 2a - 2\,, \qquad a \geq 4 \quad (6.33)$$

where we have used (6.29). Finally, consider $k = 3$ and set $\mathcal{D} = 3\sigma|_{K3} + aF$. Then

$$(3\sigma|_{K3} + aF) \cdot \sigma|_{K3} = a - 6\,. \quad (6.34)$$

Hence, for $0 \leq a \leq 5$, $\mathcal{D} \cdot \sigma|_{K3} < 0$ and equation (6.23) implies that

$$h^0(K3, \mathcal{O}_{K3}(3\sigma|_{K3} + aF)) = h^0(K3, \mathcal{O}_{K3}(2\sigma|_{K3} + aF))\,, \qquad 0 \leq a \leq 5\,. \quad (6.35)$$

For $a \geq 6$, $\mathcal{D} \cdot \sigma|_{K3} \geq 0$ and

$$(2\sigma|_{K3} + aF) \cdot \sigma|_{K3} = a - 4 > 0\,, \qquad (2\sigma|_{K3} + aF) \cdot F = 2 > 0\,. \quad (6.36)$$

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Then, equation (6.27) is applicable and gives

$$h^0(K3, \mathcal{O}_{K3}(3\sigma|_{K3} + aF)) = h^0(K3, \mathcal{O}_{K3}(2\sigma|_{K3} + aF)) + a - 5\,, \qquad a \geq 6\,. \tag{6.37}$$

To get the required information about the coefficients B and C, it is sufficient to look at one particular value of a, namely $a = 4$. In this case, expressions (6.33) and (6.35) tell us that

$$h^0(K3, \mathcal{O}_{K3}(3\sigma|_{K3} + aF)) = 6\,. \tag{6.38}$$

By using (6.10), (6.11), (6.16), and (6.38), we conclude

$$h^0(\mathcal{E}, \pi_*\mathcal{L}_a) = h^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(4) \oplus \mathcal{O}_{\mathcal{E}}(4+B) \oplus \mathcal{O}_{\mathcal{E}}(4+C)) = 6\,. \tag{6.39}$$

Since the line bundle $\mathcal{O}_{\mathcal{E}}(4)$ already has 5 sections, one of the other two line bundles has to be trivial and one has to be negative. Without loss of generality, one can choose

$$B = -4\,, \qquad C \leq -5 \tag{6.40}$$

as the solution of equation (6.39). Combining this result with (6.20), we see that

$$B = -4\,, \qquad C = -6\,. \tag{6.41}$$

Putting everything together, we conclude that for $n = 3$ and arbitrary coefficient a, the push-forward of line bundle $\mathcal{L}_a$ onto curve $\mathcal{E}$ and, therefore, onto the curve $\pi^*\mathcal{E} \cdot \sigma \subset K3$ is given by the rank three vector bundle

$$\pi_*\mathcal{L}_a = \mathcal{O}_{\mathcal{E}}(a) \oplus \mathcal{O}_{\mathcal{E}}(a-4) \oplus \mathcal{O}_{\mathcal{E}}(a-6)\,. \tag{6.42}$$

The sections of this vector bundle, specified by

$$h^0(\mathcal{E}, \pi_*\mathcal{L}_a) = 3a - 7 \tag{6.43}$$

are precisely the transition moduli of the associated small instanton transition. It is useful at this point to give a concrete example.

Example. Consider the small instanton transition presented in Example 1 earlier in this section. Recall that the pre-transition vector bundle V was specified by $B = \mathbb{F}_1$, $G = \mathrm{SU}(3)$, $a = 7$, $b = 12$ and $\lambda = 1/2$. In this case, the $M5$-brane involved in the transition is wrapped on the curve

$$z = \mathcal{E}\,. \tag{6.44}$$

Then, from (6.42) we find that the transition moduli are the holomorphic sections of the rank 3 vector bundle

$$\pi_*\mathcal{L}_7 = \mathcal{O}_{\mathcal{E}}(7) \oplus \mathcal{O}_{\mathcal{E}}(3) \oplus \mathcal{O}_{\mathcal{E}}(1) \tag{6.45}$$

over $z = \mathcal{E}$, where, from (6.1)

$$\mathcal{L}_7 = \mathcal{O}_{K3}(3\sigma|_{K3} + 7F)\,. \tag{6.46}$$

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It follows from (6.43) that the number of transition moduli is

$$h^0(\mathcal{E}, \pi_* \mathcal{L}_7) = 14\,, \tag{6.47}$$

in agreement with (5.10) and (5.96).

Now consider the case $z = \mathcal{S}$. In the previous section, we showed that the transition moduli are given by the number of global sections of the line bundle $L'|_{\pi^*\mathcal{S}}$ specified in (5.67) and (5.68). As before, denote

$$\mathcal{L}_0 = \mathcal{O}_{\pi^*\mathcal{S}}(n\sigma|_{\pi^*\mathcal{S}}) \tag{6.48}$$

and

$$\mathcal{L}_\gamma = \mathcal{O}_{\pi^*\mathcal{S}}(n\sigma|_{\pi^*\mathcal{S}} + \gamma F)\,, \qquad \gamma = b - (a+1)r\,. \tag{6.49}$$

The line bundle $\mathcal{L}_\gamma$ can be written as

$$\mathcal{L}_\gamma = \mathcal{O}_{\pi^*\mathcal{S}}(n\sigma|_{\pi^*\mathcal{S}}) \otimes \pi^* \mathcal{O}_{\mathcal{E}}(\gamma) \tag{6.50}$$

and, by using projection formula (6.5), we find that its direct image is given by

$$\pi_* \mathcal{L}_\gamma = \pi_* \mathcal{L}_0 \otimes \mathcal{O}_{\mathcal{S}}(\gamma)\,. \tag{6.51}$$

Note that since $\mathcal{C}'|_{\pi^*\mathcal{S}}$ is an n-fold cover of the base curve $\mathcal{S}$, the push-forward of any line bundle of $\mathcal{C}'|_{\pi^*\mathcal{S}}$ onto $\mathcal{S}$ is a rank n vector bundle.

At this point, for concreteness, we specify

$$n = 3\,, \qquad r = 1\,, \tag{6.52}$$

but leave coefficients a and b arbitrary. Our remarks, however, will remain true for any values of n and r. We remind the reader that when $r = 1$, the surface $\pi^*\mathcal{S}$ is the del Pezzo surface dP_9.

Since every line bundle on $\mathcal{S} = \mathbb{P}^1$ is a sum of line bundles, rank 3 vector bundle $\pi_*\mathcal{L}_0$ can always be written as

$$\pi_* \mathcal{L}_0 = \mathcal{O}_{\mathcal{S}}(A) \oplus \mathcal{O}_{\mathcal{S}}(B) \oplus \mathcal{O}_{\mathcal{S}}(C) \tag{6.53}$$

for some integers A, B and C. Hence, from (6.51)

$$\pi_* \mathcal{L}_\gamma = \mathcal{O}_{\mathcal{S}}(A+\gamma) \oplus \mathcal{O}_{\mathcal{S}}(B+\gamma) \oplus \mathcal{O}_{\mathcal{S}}(C+\gamma)\,. \tag{6.54}$$

We must now compute the values of A, B and C. One can show, using a Leray spectral sequence, that for any values of n and r

$$H^0(\mathcal{S}, \pi_* \mathcal{L}_\gamma) = H^0(\pi^*\mathcal{S}, \mathcal{L}_\gamma)\,. \tag{6.55}$$

Here, we will continue to assume that $n = 3$ and $r = 1$. Now, take $\gamma = 0$. Then the bundle $\mathcal{L}_\gamma$ is equal to $\mathcal{L}_0 = \mathcal{O}_{dP_9}(n\sigma|_{dP_9})$. Since for $r = 1$ one can show, using (3.23) and (3.35), that

$$\sigma|_{dP_9} \cdot \sigma|_{dP_9} = -1\,, \tag{6.56}$$

the bundle $\mathcal{O}_{dP_9}(n\sigma|_{dP_9})$ has only one section and, hence,

$$h^0(\mathcal{S}, \pi_*\mathcal{L}_0) = h^0(dP_9, \mathcal{L}_0) = 1\,, \tag{6.57}$$

where we have used (6.55). From equations (6.14) and (6.15) we conclude that

$$A = 0\,, \qquad B, C < 0\,. \tag{6.58}$$

Next, consider the case $\gamma > 2$ that corresponds, according to (3.37), to the case when the spectral cover is chosen to be positive. From (5.91) and (6.55) we easily find that

$$h^0(S, \pi_*\mathcal{L}_\gamma) = 3\gamma - 2\,. \tag{6.59}$$

As in the previous example, one must have

$$\gamma + B \geq 0\,, \qquad \gamma + C \geq 0\,, \tag{6.60}$$

which implies, using (6.14) and (6.54), that

$$h^0(S, \pi_*\mathcal{L}_\gamma) = (\gamma + 1) + (\gamma + B + 1) + (\gamma + C + 1)\,. \tag{6.61}$$

But, from (6.59), this expression must equal $3\gamma - 2$. Therefore

$$B + C = -5\,. \tag{6.62}$$

To find the values of B and C we use the same techniques as in the case $z = \mathcal{E}$. For any divisor $\mathcal{D}$ in dP_9 we have the following short exact sequence

$$0 \to \mathcal{O}_{dP_9}(\mathcal{D} - \sigma|_{dP_9}) \to \mathcal{O}_{dP_9}(\mathcal{D}) \to \mathcal{O}_{\sigma|_{dP_9}}(\mathcal{D} \cdot \sigma|_{dP_9}) \to 0\,, \tag{6.63}$$

implying the long exact sequence of cohomology groups given by

$$\begin{aligned} 0 &\to H^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D} - \sigma|_{dP_9})) \to H^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D})) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D} \cdot \sigma|_{dP_9})) \to \\ &\to H^1(dP_9, \mathcal{O}_{dP_9}(\mathcal{D} - \sigma|_{dP_9})) \to\ . \end{aligned} \tag{6.64}$$

First, assume that $\mathcal{D} \cdot \sigma|_{dP_9} < 0$. Then, it follows from (6.15) that $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D} \cdot \sigma|_{dP_9})) = 0$ and, hence

$$h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D})) = h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D} - \sigma|_{dP_9}))\,, \qquad \mathcal{D} \cdot \sigma|_{dP_9} < 0\,. \tag{6.65}$$

Now assume $\mathcal{D} \cdot \sigma|_{dP_9} \geq 0$. Equation (6.14) then implies

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D} \cdot \sigma|_{dP_9})) = \mathcal{D} \cdot \sigma|_{dP_9} + 1\,. \tag{6.66}$$

Note, using $\sigma|^2_{dP_9} = -1$, that $(\mathcal{D} - \sigma|_{dP_9}) \cdot \sigma|_{dP_9} > 0$. If we further assume that $\mathcal{D} \cdot F > 1$, then $(\mathcal{D} - \sigma|_{dP_9}) \cdot F > 0$ and, hence, $\mathcal{D} - \sigma|_{dP_9}$ is a positive divisor in dP_9. It then follows from the Kodaira vanishing theorem that

$$H^q(dP_9, \mathcal{O}_{dP_9}(\mathcal{D} - \sigma|_{dP_9})) = 0 \tag{6.67}$$

for $q > 0$. Hence, the exact sequence (6.64) truncates to

$$0 \to H^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D}-\sigma|_{dP_9})) \to H^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D})) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{D}\cdot\sigma|_{dP_9})) \to 0\,. \quad (6.68)$$

Therefore, using (6.66), for $\mathcal{D} - \sigma|_{dP_9} > 0$ we have

$$h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D})) - h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{D} - \sigma|_{dP_9})) = \mathcal{D} \cdot \sigma|_{dP_9} + 1\,, \qquad \mathcal{D} \cdot \sigma|_{dP_9} \geq 0\,. \quad (6.69)$$

Now, from equations (6.65) and (6.69) we can compute $h^0(dP_9, \mathcal{O}_{dP_9}(k\sigma|_{dP_9} + \gamma F))$ for $k = 0, 1, 2, 3$. Start with $k = 0$. In this case, equations (6.55) and (6.14) imply that

$$h^0(dP_9, \mathcal{O}_{dP_9}(\gamma F)) = h^0(dP_9, \pi^*\mathcal{O}_{\mathcal{S}}(\gamma)) = h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(\gamma)) = \gamma + 1\,, \qquad \gamma \geq 0\,. \quad (6.70)$$

Now take $k = 1$. By the same argument as in the previous case, the only freedom to move a divisor of the class $\sigma|_{dP_9} + \gamma F$ is in the fibers, so we conclude

$$h^0(dP_9, \mathcal{O}_{dP_9}(\sigma|_{dP_9} + \gamma F)) = h^0(dP_9, \mathcal{O}_{dP_9}(\gamma F)) = \gamma + 1\,, \qquad \gamma \geq 0\,. \quad (6.71)$$

As the next step, take $k = 2$ and set $\mathcal{D} = 2\sigma|_{dP_9} + \gamma F$. First, note that

$$(2\sigma|_{dP_9} + \gamma F) \cdot \sigma|_{dP_9} = \gamma - 2\,. \quad (6.72)$$

This breaks into two cases. For $\gamma = 0, 1$, $\mathcal{D} \cdot \sigma|_{dP_9} < 0$. Then, it follows from (6.65) and (6.71) that

$$h^0(dP_9, \mathcal{O}_{dP_9}(2\sigma|_{dP_9} + \gamma F)) = h^0(dP_9, \mathcal{O}_{dP_9}(\sigma|_{dP_9} + \gamma F)) = \gamma + 1\,, \qquad \gamma = 0, 1\,. \quad (6.73)$$

Now let $\gamma \geq 2$. In this case, $\mathcal{D} \cdot \sigma|_{dP_9} \geq 0$ and

$$(\sigma|_{dP_9} + \gamma F) \cdot \sigma|_{dP_9} = \gamma - 1 > 0\,, \qquad (\sigma|_{dP_9} + \gamma F) \cdot F = 1 > 0\,. \quad (6.74)$$

Then equation (6.69) is applicable and implies that

$$h^0(dP_9, \mathcal{O}_{dP_9}(2\sigma|_{dP_9} + \gamma F)) = h^0(dP_9, \mathcal{O}_{dP_9}(\sigma|_{dP_9} + \gamma F)) + \gamma - 1\,, \qquad \gamma \geq 1\,, \quad (6.75)$$

where we have used (6.71). Finally, consider $k = 3$ and set $\mathcal{D} = 3\sigma|_{dP_9} + \gamma F$. Then

$$(3\sigma|_{dP_9} + \gamma F) \cdot \sigma|_{dP-9} = \gamma - 3\,. \quad (6.76)$$

Hence, for $0 \leq \gamma \leq 2$, $\mathcal{D} \cdot \sigma|_{dP_9} < 0$ and equation (6.65) implies

$$h^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + \gamma F)) = h^0(dP_9, \mathcal{O}_{dP_9}(2\sigma|_{dP_9} + \gamma F))\,, \qquad 0 \leq \gamma \leq 2\,. \quad (6.77)$$

For $\gamma \geq 3$, $\mathcal{D} \cdot \sigma|_{dP_9} \geq 0$ and

$$(2\sigma|_{dP_9} + \gamma F) \cdot \sigma|_{dP_9} = \gamma - 2 > 0\,, \qquad (2\sigma|_{dP_9} + \gamma F) \cdot F = 2 > 0\,. \quad (6.78)$$

Then, equation (6.69) is applicable and gives

$$h^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + \gamma F)) = h^0(dP_9, \mathcal{O}_{dP_9}(2\sigma|_{dP_9} + a\gamma F)) + \gamma - 2\,, \qquad \gamma \geq 2\,. \quad (6.79)$$

To get the required information about the coefficients B and C, it is sufficient to look at one particular value of γ, namely $\gamma = 4$. In this case, expressions (6.75) and (6.77) tell us that

$$h^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + \gamma F)) = 4\,. \tag{6.80}$$

By using (6.54), (6.55), (6.58), and (6.80), we conclude

$$h^0(\mathcal{S}, \pi_* \mathcal{L}_\gamma) = h^0(\mathcal{S}, \mathcal{O}_\mathcal{S}(2) \oplus \mathcal{O}_\mathcal{S}(2+B) \oplus \mathcal{O}_\mathcal{S}(2+C)) = 4\,. \tag{6.81}$$

Since the line bundle $\mathcal{O}_\mathcal{E}(2)$ already has 3 sections, one of the other two line bundles has to be trivial and one has to be negative. Without loss of generality, one can choose

$$B = -2\,, \qquad C \leq -3 \tag{6.82}$$

as the solution of equation (6.81). Combining this result with (6.62), we see that

$$B = -2\,, \qquad C = -3\,. \tag{6.83}$$

Putting everything together, we conclude that, for $n = 3$, $r = 1$ and arbitrary coefficients a and b, one has

$$\pi_* \mathcal{L}_\gamma = \mathcal{O}_\mathcal{S}(b - (a+1)) \oplus \mathcal{O}_\mathcal{S}(b - (a+3)) \oplus \mathcal{O}_\mathcal{S}(b - (a+4))\,. \tag{6.84}$$

The holomorphic sections of this vector bundle, specified by

$$h^0(\mathcal{S}, \pi_* \mathcal{L}_\gamma) = 3(b - (a+1)) - 2 \tag{6.85}$$

are in one-to-one correspondence with the small instanton transition moduli. It is useful at this point to give a concrete example.

Example. Consider the small instanton transition presented in Example 2 earlier in this section. Recall that the pre-transition vector bundle V was specified by $B = \mathbb{F}_1$, $G = \mathrm{SU}(3)$, $a = 7$, $b = 12$ and $\lambda = 1/2$. In this example, the $M5$-brane involved in the transition is wrapped on the curve

$$z = \mathcal{S}\,. \tag{6.86}$$

Then, from (6.84) we find that the transition moduli are the holomorphic sections of the rank 3 vector bundle

$$\pi_* \mathcal{L}_4 = \mathcal{O}_\mathcal{S}(4) \oplus \mathcal{O}_\mathcal{S}(2) \oplus \mathcal{O}_\mathcal{S}(1) \tag{6.87}$$

over $z = \mathcal{S}$, where, from (6.49)

$$\mathcal{L}_4 = \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F)\,. \tag{6.88}$$

It follows from (6.85) that the number of transition moduli is

$$h^0(\mathcal{S}, \pi_* \mathcal{L}_4) = 10\,, \tag{6.89}$$

in agreement with (5.12) and (5.99).

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The above method of finding the push-forwards of the line bundles $\mathcal{L}_a$ and $\mathcal{L}_\gamma$ onto their respective curves z is rather technical. It is useful, therefore, to give a more intuitive, algebraic derivation of the rank 3 vector bundles (6.42) and (6.84). To do this, recall from section 3 that the Calabi-Yau threefold X is given by the Weierstrass equation (3.1), with x, y and z being sections of the following line bundles

$$x \sim \mathcal{O}_P(1) \otimes \mathcal{L}^2\,, \qquad y \sim \mathcal{O}_P(1) \otimes \mathcal{L}^3\,, \qquad z \sim \mathcal{O}_P(1)\,. \tag{6.90}$$

First consider $z = \mathcal{E}$ and let P' denote the restriction of the $\mathbb{CP}^2$-bundle P to the curve $\mathcal{E}$. Then the elliptically fibered surface $\pi^*\mathcal{E} = K3$ is defined by a similar Weierstrass equation, but now as a divisor in P'. We find the following identifications

$$\mathcal{L}|_{K3} = \mathcal{O}_{K3}(2F)\,, \qquad \mathcal{O}_P(1)|_{K3} = \mathcal{O}_{K3}(3\sigma|_{K3})\,. \tag{6.91}$$

The first identification follows from the definition of $\mathcal{L}$ as the pullback to P of the conormal bundle to $\sigma(B)$ in X. To see this, note that when we restrict $\mathcal{L}$ to $\sigma|_{K3}$, we get the conormal bundle to $\sigma|_{K3}$ in $K3$, which is $\mathcal{O}_{\sigma|_{K3}}(2)$. Pulling this bundle back to $K3$ gives $\mathcal{O}_{K3}(2F)$. The second identification follows from the fact that $\mathcal{O}_P(1) = \mathcal{O}_X(3\sigma)$. Using equations (6.90) and (6.91), we see from the Weierstrass equation defining $K3$ that x, y and z are sections of the line bundles

$$x \sim \mathcal{O}_{K3}(3\sigma|_{K3} + 4F)\,, \qquad y \sim \mathcal{O}_{K3}(3\sigma|_{K3} + 6F)\,, \qquad z \sim \mathcal{O}_{K3}(3\sigma|_{K3})\,. \tag{6.92}$$

Recall that our problem is to find the push-forward of the line bundle $\mathcal{L}_a = \mathcal{O}_{K3}(3\sigma|_{K3} + aF)$. Denote by s_k a section of $H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(k))$. Using a Leray spectral sequence, one can show that

$$H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(k)) = H^0(K3, \mathcal{O}_{K3}(kF))\,. \tag{6.93}$$

Hence, s_k lifts to a section of $H^0(K3, \mathcal{O}_{K3}(kF))$, which we will also denote by s_k. Clearly, one can construct the following sections of $\mathcal{L}_a$

$$s_a z + s_{a-4} x + s_{a-6} y\,. \tag{6.94}$$

This shows explicitly how sections of $H^0(K3, \mathcal{L}_a)$ can be written as a sum of three terms involving sections of $H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(a))$, $H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(a-4))$ and $H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(a-6))$ respectively. We conclude that

$$\pi_*\mathcal{L}_a = \mathcal{O}_\mathcal{E}(a) \oplus \mathcal{O}_\mathcal{E}(a-4) \oplus \mathcal{O}_\mathcal{E}(a-6)\,, \tag{6.95}$$

exactly as in (6.42).

A similar argument applies to the case $z = \mathcal{S}$. Let P'' denote the restriction of the bundle P to the curve $\mathcal{S}$. The Weierstrass equation (3.1) now defines the elliptic surface dP_9 as a divisor in P''. Since $\sigma|_{dP_9} \cdot \sigma|_{dP_9} = -1$, the conormal bundle to $\sigma|_{dP_9}$ in dP_9 is $\mathcal{O}_{\sigma|_{dP_9}}(1)$. As a result

$$\mathcal{L}|_{dP_9} = \mathcal{O}_{dP_9}(F)\,, \qquad \mathcal{O}_P(1)|_{dP_9} = \mathcal{O}_{dP_9}(3\sigma|_{dP_9})\,. \tag{6.96}$$

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Using this and equation (6.90), we find that x, y and z are now sections of the following line bundles on dP_9

$$x \sim \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 2F)\,, \qquad y \sim \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 3F)\,, \qquad z \sim \mathcal{O}_{dP_9}(3\sigma|_{dP_9})\,. \tag{6.97}$$

If s_k denotes a section of $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(k)) = H^0(dP_9, \mathcal{O}_{dP_9}(kF))$, we can construct the following sections of the line bundle $\mathcal{L}_\gamma = \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + \gamma F)$

$$s_\gamma z + s_{\gamma-2} x + s_{\gamma-3} y\,. \tag{6.98}$$

This shows how sections of $H^0(dP_9, \mathcal{L}_\gamma)$ can be written as a sum of three terms involving sections of $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(\gamma))$, $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(\gamma-2))$ and $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(\gamma-3))$ respectively. We conclude that

$$\pi_* \mathcal{L}_\gamma = \mathcal{O}_{\mathcal{S}}(\gamma) \oplus \mathcal{O}_{\mathcal{S}}(\gamma - 2) \oplus \mathcal{O}_{\mathcal{S}}(\gamma - 3)\,, \tag{6.99}$$

exactly as in (6.84).

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